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Non-critical superstrings from four-dimensional gauge theory

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Abstract: We study large-N double-scaling limits of U(N) gauge theories in four dimensions. We focus on theories in a partially confining phase where an abelian subgroup $\hat{G}$ of the gauge group remains unconfined. Double-scaling is defined near critical points in the parameter/moduli space where states charged under $\hat{G}$ become massless. In specific cases, we present evidence that the double-scaled theory is dual to a non-critical superstring background. Models studied include the β-deformation of $\mathcal{N} = 4$ SUSY Yang-Mills which leads to a non-critical string theory with sixteen supercharges. We also study $\mathcal{N} = 1$ SUSY Yang-Mills theory coupled to a single chiral superfield with a polynomial superpotential which leads to a related string theory with eight supercharges. In both cases the string coupling is small and the background is free from Ramond-Ramond flux.

Keywords: 1/N Expansion, Topological Strings, Supersymmetry and Duality.

Contents

1. Introduction and overview

The phenomenon of confinement plays a key role in the physics of the strong interactions. Obtaining an analytic description of the confining phase is an important goal which has, so far, remained elusive. One of the best hopes for progress in this direction is the large-N

expansion of SU(N) gauge theory suggested many years ago by 't Hooft [1]. In particular, pure Yang-Mills theory is believed to be dual to a weakly-coupled string theory in this limit. The large-N spectrum consists of infinite towers of stable glueballs corresponding to the excitations of a closed string. The AdS/CFT correspondence [2] provides examples of confining models [3, 29, 4, 5] where the dual theory is a compactification of critical superstring theory in ten dimensions. Generally, however, the dual background contains Ramond-Ramond (RR) flux or there are other obstacles to quantization of the string. In these cases, progress is only possible in the supergravity limit where all but the lightest glueball states decouple.[1] In this paper we will propose limits of certain confining theories which bypass this problem and lead to a weakly-coupled dual string theory in a pure Neveu-Schwarz (NS) background. The resulting backgrounds can also be understood as non-critical superstring theories of the type first introduced in [7]. Our analysis involves a large-N *double-scaling limit* closely related to those arising in the study of matrix models and also to the double-scaling limits of Little String Theory [9, 8]. In the rest of this introductory section we will outline the main ideas and results.

In the usual 't Hooft large-N expansion, the leading order consists of an infinite sum of planar Feynman diagrams. Using the double-line notation, individual diagrams in this sum can have an arbitrary number of "holes" corresponding to closed index loops. In string theory constructions, where SU(N) gauge theory describes the interactions of open strings living on the worldvolume of N coincident D-branes, these holes are simply the boundaries of the string world sheet which arise naturally in open string perturbation theory. The basic idea of large-N duality is that, by summing over all such open string diagrams, a dual description in terms of closed strings is obtained. In spacetime, this amounts to replacing the D-branes by their near-horizon geometry which includes N units of RR flux. In a large-radius limit where the RR field-strength is small, we find that the holes have effectively been filled in to give a continuous closed string world-sheet. What happens away from this limit is less clear, as we do not know how to quantize the string. Recent work [10, 11] offers interesting hints that holes reappear in the closed string description as "bubbles" of a new phase of the worldsheet theory and that they are directly related to the presence of RR flux. Another line of development, reviewed in [12], suggests that the correct effective description of the world-sheet in the presence of strong RR fields is actually discrete rather than continuous.[2] For the basic case of the duality between string theory $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SUSY Yang-Mills the continuous string is effectively replaced by a discrete spin chain when the background RR fields become large [13]. This discretisation also suggests the presence of world-sheet holes.

Our main goal here is to find four-dimensional examples where the problems of RR flux and of the associated holes in the string worldsheet can be avoided. It is illuminating to begin by recalling the duality between matrix models and $c \leq 1$ non-critical string theory (for a review see [14]). The matrix model has critical points in its parameter space

[1] For some models the spectrum can also be calculated for certain limits of large quantum numbers [6].

[2] Such an effective description might emerge when a more traditional continuous description of the worldsheet, like that suggested in [10, 11], becomes strongly coupled. Further discussion of these points is given in section 8.

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where the sum over diagrams of each fixed genus diverges in a characteristic way and the standard $1/N$ expansion breaks down. Instead one may define an alternative double-scaling limit where the couplings are tuned to their critical values as $N \to \infty$ to obtain a finite contribution from each genus. The resulting genus expansion is controlled by a parameter $1/N_{\rm eff}$ which is held fixed in the limit. In this context, the Feynman diagrams have a direct interpretation as discretisations of a dual string world-sheet. As we approach the critical point, the sum is dominated by large diagrams with many holes. After an appropriate rescaling, we obtain a continuum limit on a worldsheet of fixed size, where the average size of the holes goes to zero. The result is a continuum string propagating in a linear dilaton background. The strong-coupling region is removed by potential terms in the world-sheet action and the effective string coupling is identified with the double-scaling parameter $1/N_{\rm eff}$.

In this paper we will define analogous double-scaling limits in four dimensions and argue that they lead to pure NS backgrounds. We will make contact with several features of the matrix model case, including the divergence in the sum of planar diagrams, the appearance of a linear dilaton in the dual string theory and the emergence of universality classes. More generally, our results are consistent with an interpretation of double-scaling as a continuum limit on the string world-sheet. The fact that such a limit yields a pure NS background also fits well with the association between RR flux and world-sheet holes mentioned above.

As in the case of matrix models, the first step is to find an appropriate critical point. The examples we will study here involve four-dimensional confining gauge theories in a phase where an abelian subgroup of the gauge group remains unconfined and there is no mass gap. One well-studied model [21, 24] which exhibits this behaviour is $\mathcal{N} = 1$ SUSY Yang-Mills with gauge group $G = \mathrm{U}(N)$ coupled to an adjoint chiral multiplet Φ with a polynomial superpotential,[3]

$$W = \varepsilon \mathrm{Tr}_N \left[\frac{\Phi^{m+1}}{m+1} - \sum_{l=1}^{m} g_l \frac{\Phi^l}{l} \right] \tag{1.1}$$

of degree $m + 1$. In generic vacua, the $\mathrm{U}(N)$ gauge symmetry is confined down to a $\mathrm{U}(1)^m$ subgroup denoted $\hat{G}$. The spectrum contains m massless photons and their $\mathcal{N} = 1$ superpartners. Despite the absence of a mass gap, the Wilson loop evaluated in the fundamental representation obeys an area law as it does in more conventional confining phases. We will also study the β-deformation of $\mathcal{N} = 4$ SUSY Yang-Mills which has a partially confining phase of exactly the same type [26, 27]. Beyond these specific models, vacua in this phase are generic to all $\mathcal{N} = 1$ SUSY gauge theories coupled to adjoint matter with a superpotential. Such vacua also occur in non-supersymmetric models with adjoint scalars. The phenomena described below should have interesting generalisations to this wider class of models.

To explain the relevance of these partially confining models, we begin by describing some common features of their physics which will be demonstrated in section 3 below.

[3]This model is non-renormalisable and we will work with a fixed UV cut-off M_{UV} in place.

First, as in pure Yang-Mills theory, the large-N spectrum includes a tower of weakly-interacting glueballs corresponding to excitations of a dual closed string. These states are neutral under the unconfined gauge group $\hat{G}$ and have masses which remain fixed as $N \to \infty$. An important new ingredient is that these models also contain states which carry electric and/or magnetic charges under $\hat{G}$. For reasons which will be explained in section 3, we will refer to these states collectively as dibaryons. In the dual string theory, these dibaryons correspond to wrapped/stretched D-branes. In generic vacua their masses grow linearly with N in accord with the identification $g_s \sim 1/N$. However, by varying the parameters/moduli we can also find special vacua where some dibaryons become massless. These are the critical points we will study. Near these special points standard large-N counting arguments show that the 't Hooft $1/N$ expansion breaks down. For the model (1.1), we will relate this directly to a divergence in the sum of planar diagrams. These critical points can also be related to the occurrence of singularities in the dual string background where wrapped/stretched D-branes become massless. In the β-deformed theory, we will see that the region near the singular point in spacetime is replaced by an infinite throat with a linear dilaton. The breakdown of closed string perturbation theory caused by the growth of the dilaton corresponds to the breakdown of the $1/N$ expansion in the dual field theory.

As in the case of zero-dimensional matrix models, we can attempt to define a double-scaling limit which yields a finite effective string coupling. In field theory, this corresponds to a limit where couplings are tuned to their critical values as $N \to \infty$ in such a way that the mass M_B of the lightest dibaryon remains constant. The tension T of the confining string, which sets the glueball mass is also held fixed. As noted above, the conventional 't Hooft large-N limit leads to a free theory of colour singlet states with all interactions suppressed by powers of $1/N$. The key characteristic of the double-scaling limit studied here is the emergence of a decoupled sector of states with residual interactions controlled instead by the double scaling parameter $1/N_{\rm eff} \sim \sqrt{T}/M_B$. For both models studied in detail in this paper, we will argue that this interacting sector has a tractable dual description as a double-scaled Little String Theory or, via holography [15, 16], as a non-critical superstring theory. Apart from providing, in these special cases, a solvable limit of confining gauge theory, these results also suggest a new approach to the non-perturbative definition of string theory in linear dilaton backgrounds.

In the case of the β-deformed theory, a double-scaling limit of exactly the type described above was considered in [27]. In the following we will review the relevant results of [27] adding several new details and physical interpretation. We will then go on to describe our new results for the model with superpotential (1.1).

1.1 Review of the β-deformation

The β-deformation of $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group SU(N) is dual to a certain deformation of IIB string theory on $AdS_5 \times S^5$ [17, 18]. The model has a partially confining branch for special values of the deformation parameter. Generic vacua on this branch correspond to configurations with NS fivebranes located at different radial positions in AdS_5. Each NS5 is also wrapped on a (topologically trivial) toroidal submanifold of S^5.

The general features of the large-N spectrum described above are visible in the string dual. Electric (magnetic) dibaryons are dual to toroidally-wrapped D3 branes (D-strings), stretched between the NS5 branes. Critical points with massless dibaryons correspond to configurations with coincident NS fivebranes. In these vacua, the theory flows to $\mathcal{N} = 4$ SUSY Yang-Mills in the IR, with gauge group SU(m) for m coincident fivebranes.

For sufficiently large 't Hooft coupling, the critical points described above can be studied using supergravity. Adapting earlier work by Polchinski and Strassler [29], the supergravity dual was found in [27]. The dual geometry interpolates smoothly between an asymptotically AdS region in the UV and the near-horizon geometry of coincident NS5 branes wrapped on T^2. The wrapped NS5 branes also carry a total of N units of D3 brane charge on their worldvolume. As usual the near-horizon region includes a semi-infinite throat with a linear dilaton. Although the full background includes non-zero Ramond-Ramond fields, these die off rapidly as we enter the throat leaving only the fields of the standard NS fivebrane solution.

The singularity corresponding to coincident IIB NS5 branes in asymptotically flat ten-dimensional spacetime has been studied from many points of view. In particular, the system has a decoupling limit leading to a six-dimensional Little String Theory (LST) [9]. In this limit, the asymptotically flat region of the solution decouples leaving a throat which is infinite in both directions. The IIB theory on this infinite throat background is holographically dual to the Little String Theory living on the fivebranes [15]. In the present case we have coincident NS5 branes embedded in an asymptotically AdS spacetime. The novelty here is that the *full* geometry, including both the asymptotic region and the fivebranes, is holographically dual to the four-dimensional gauge theory we started with. What remains is to reinterpret known aspects of fivebrane physics in terms of the four-dimensional theory.

The large-N expansion breaks down at critical points where dibaryons become massless. This is directly visible in the string dual where the growth of the dilaton leads to the breakdown of closed string perturbation theory. However, as for other linear dilaton backgrounds, we can cure this problem by deforming the theory in a way which eliminates the strong coupling region. For NS5 branes in flat space, such a deformation was identified by Giveon and Kutasov [8]. These authors argued that separating the fivebranes slightly in their transverse directions has the effect of replacing the infinite tube in the string dual with a semi-infinite cigar where the dilaton reaches a maximum value at the tip of the cigar. More precisely, they consider a double-scaling limit where the separation r_0, between the fivebranes and the asymptotic string coupling, g_s, both go to zero with the ratio r_0/g_s and α' held fixed. This ensures that the masses $M_{D1} \sim r_0/g_s\alpha'$ of stretched D-strings remain constant in the limit. The maximum value of the effective string coupling attained at the tip of the cigar is then $g_{\rm cigar} \sim 1/M_{D1}\sqrt{\alpha'}$.

Following [27], we can now reinterpret the double-scaling limit of Giveon and Kutasov in the dual gauge theory. As usual taking the limit $g_s \to 0$ with α' fixed corresponds to a 't Hooft large-N limit. The new ingredient is that the moduli of the partially confining branch (corresponding to the separation of the fivebranes) are scaled so as to keep the masses of the lightest electric and magnetic dibaryons fixed. In this limit, the asymptotically AdS region of the geometry is separated from the region near the tip of the cigar by a throat of length $\sim \log N$. In the dual field theory this corresponds to the decoupling of two different sectors of the large-N Hilbert space. One sector, corresponding to states localised near the tip of the cigar, consists of infinite towers of glueballs with residual interactions controlled by the coupling $1/N_{\rm eff} = g_{\rm cigar}$. The formula for this coupling and the effective string tension in terms of field theory parameters is given in eqs. (7.7), (7.8) below. When $1/N_{\rm eff}$ is small, this sector of the theory is described by tree-level string theory on an exactly solvable background. This in turn is equivalent to a six-dimensional double-scaled LST with sixteen supercharges compactified on T^2. The emergence of two additional compact directions in the dual field theory is related to the phenomenon of deconstruction [27]. The other sector of the theory, corresponding to the asymptotically AdS region, becomes free as $N \to \infty$.

The type of decoupling described above is quite novel from a field theoretic point of view. Normally two sectors of states decouple only when there is a large hierarchy between the corresponding mass scales. We emphasize that there is no such hierarchy in the present case: the masses of typical states in the two sectors are essentially the same. The two sectors decouple because they are separated by an infinitely long throat in the dual geometry. The decoupling is therefore related to the approximate locality of the bulk theory which is usually hard to understand from the boundary side of the correspondence. An obvious question is whether the phenomena described above are somehow special to the β-deformation of $\mathcal{N} = 4$ SUSY Yang-Mills. A more interesting possibility is that non-trivial double-scaling limits can be defined for other partially confining theories which have critical points in their parameter/moduli spaces. In this paper, we will show that the model with superpotential (1.1) provides an interesting new example of this type.

1.2 A new example

The $\mathcal{N} = 1$ theory with superpotential (1.1) is dual to Type II string theory on a certain non-compact Calabi-Yau manifold with non-zero RR fluxes [32]. In this context, the electric and magnetic dibaryons of the field theory correspond to D-branes wrapped on particular topologically non-trivial cycles which each have zero units of RR flux. At a critical point, one or more of these cycles shrink to zero size, leading to a singular geometry, and the corresponding dibaryons become massless. Although various singularities are possible we focus on a particular case which has a straightforward interpretation in the dual field theory. The singularity occurs when the F-term couplings take the values $g_1 = 2\Lambda^m$, with $g_l = 0$ for $l > 1$ [43, 44]. Here Λ is the RG invariant scale of the gauge theory. In this case, the dual Calabi-Yau is described by the polynomial equation,

$$u^2 + v^2 + w^2 + \varepsilon^2 z^m (z^m - 4\Lambda^m) = 0 \tag{1.2}$$

which exhibits a generalised conifold singularity at the origin $u = v = w = z = 0$. Unlike the β-deformed case, there is no obvious limit where the dual string geometry (1.2) can be studied using supergravity. Instead we will analyse the critical point using field theory methods.

The low-energy F-term effective action for the model (1.1) has been determined by Dijkgraaf and Vafa [21]. As we review in section 3, the resulting F-term action includes 2-, 3- and 4-point interaction vertices for certain single-trace operators which arise as components of $\mathcal{N}=1$ chiral superfields. As usual these quantities are dominated by planar Feynman diagrams in the 't Hooft large-N limit. A remarkable feature of the results of Dijkgraaf and Vafa is that they provide the first known examples where the sum over planar diagrams can be evaluated exactly in four dimensions. In fact these special F-term quantities are given exactly by a sum over planar diagrams even at finite N. The key simplification exploited by Dijkgraaf and Vafa is that the planar diagrams in question effectively reduce to those of an auxiliary zero-dimensional complex matrix model [22]. As in other examples related to matrix models, the existence of the critical point at $g_1 = 2\Lambda^m$ (with $g_l = 0$ for $l > 1$) reflects the finite radius of convergence of the sum over planar diagrams. As we approach the critical point this sum diverges. Near the critical point we define a large-N double-scaling limit, which tames this divergence, leaving a theory with finite (but non-zero) interactions between colour-singlet states of fixed mass.[4] In section 5, we argue that the unique limit with this property involves taking $N \to \infty$, $\varepsilon \to \infty$ and $\delta = g_1 - 2\Lambda^m \to 0$ with the following quantities held fixed,

$$\tilde{\varepsilon} = \frac{\varepsilon}{N} \qquad \Delta = \varepsilon\delta^{\frac{m+2}{2m}} \qquad \text{and} \qquad \tilde{g}_l = \frac{g_l}{\delta^{\frac{m-l+1}{m}}} \qquad \text{for} \qquad l = 2, \dots, m\,. \tag{1.3}$$

As in the conventional 't Hooft large-N limit, the dynamical scale Λ is also held fixed. The relation of this limit to the known double-scaling limits of the associated matrix model is discussed in section 5.

At the level of the F-term effective action, we can demonstrate some interesting and suggestive properties of the resulting double-scaled theory. In particular, we find that the theory has a sector which remains interacting in the double-scaling limit . This sector contains[5] $p = [(m-1)/2]$ massless U(1) vector multiplets of $\mathcal{N}=1$ supersymmetry and p neutral chiral multiplets. The effective superpotential for these chiral multiplets goes to zero in the double-scaling limit and, in particular, their masses vanish in this limit. The interactions between these degrees of freedom are controlled by a coupling $1/N_{\text{eff}} \sim 1/\Delta$ which is held fixed in the double-scaling limit. In contrast F-term interactions with the remaining degrees of freedom of the theory are suppressed by powers of $1/N$.

Like the case of the β-deformed theory, the critical point described above occurs when electric and magnetic dibaryons become massless. These states correspond to D-branes in the dual string theory wrapped on cycles which vanish at the critical point. The resulting divergences in the F-term action can be interpreted as a consequence of the additional massless degrees of freedom. On the other hand, the masses of these states scale like

[4]The reader may wonder why a similar analysis is not possible for the β-deformed theory described in the previous section. The answer is that, in that case, the F-term action on the partially confining branch is actually trivial because of the constraints of conformal invariance and global symmetries. To avoid confusion, note that the matrix model analysis of [52, 53, 28] was applied to the Coulomb branch of the model where the F-terms are non-trivial.

[5]Here $[k]$ denotes the greatest integer less than or equal to k.

$1/g_s \sim N$ in generic vacua away from the critical point. Hence it is natural to interpret the double-scaling limit as a limit where the couplings approach their critical values as $N \to \infty$ so that the dibaryon masses are held fixed.

On the string theory side, this again corresponds to a limit of the type considered by Giveon and Kutasov. In fact, the case of a generalised conifold singularity of the form

$$U^2 + V^2 + W^2 + Z^m = 0 \tag{1.4}$$

arising in a Type IIB Calabi-Yau geometry *without flux* was considered in [8]. The decoupling limit $g_s \to 0$, yields a Little String Theory with eight supercharges which flows to the $\mathcal{N} = 2$ Argyres-Douglas (AD) fixed-point[6] of type a_{m-1} in the IR. The resulting LST is holographically dual to an infinite throat with a linear dilaton which describes the IIB string propagating in the region near the singularity. As usual the singularity can be resolved by blowing up three-cycles so that wrapped D3 branes have non-zero masses M_{D3}. A double-scaling limit is then obtained by scaling the resolution parameters, so that the masses of these state remains fixed as $g_s \to 0$. As for the NS fivebrane theory discussed above, the effect of double-scaling is to replace the infinite throat in the holographic dual by a semi-infinite cigar. The effective string coupling then reaches a maximum value $g_{\text{cigar}} \sim 1/(M_{D3}\sqrt{\alpha'})$ at the tip of the cigar.

In the present case we are interested in the non-compact Calabi-Yau geometry (1.2) which includes n units of flux through each of m non-intersecting three-cycles. The presence of flux breaks the $\mathcal{N} = 2$ supersymmetry of the Calabi-Yau compactification down to an $\mathcal{N} = 1$ subalgebra. As described above, the dual field theory has a double-scaling limit with a decoupled sector where residual interactions are controlled by the parameter $1/N_{\text{eff}} \sim 1/\Delta$. Inspired by the example of the β-deformed theory, we would like to propose that the interacting sector is dual to the double-scaled Little String Theory which arises in the Giveon-Kutasov limit of the generalised conifold singularity without flux. As before the effective string coupling g_{cigar} is identified with the double-scaling parameter $1/N_{\text{eff}}$ and $1/(2\pi\alpha')$ is identified with the tension of the confining string. The identification of string theory and gauge parameters is given in more detail in eq. (6.23) below.

The most obvious objection to this proposal is that the double-scaled LST in question preserves eight supercharges corresponding to $\mathcal{N} = 2$ supersymmetry in four dimensions. In contrast, the gauge theory with superpotential (1.1) has only $\mathcal{N} = 1$ SUSY. Our claim is that supersymmetry is enhanced to $\mathcal{N} = 2$ only in the decoupled sector and that the enhancement occurs specifically in the double-scaling limit. We will argue that supersymmetry enhancement in the decoupled sector occurs at all energy scales not just in the IR. This is similar to the enhancement of $\mathcal{N} = 1$ SUSY to $\mathcal{N} = 4$ which occurs in the the decoupled sector of the β-deformed theory. The F-term calculations described above provide evidence in favour of this conclusion. In particular, the superpotential for the fields in the decoupled sector vanishes in the double-scaling limit and the resulting massless field content described above is consistent with the presence of $p = [(m-1)/2]$ massless vector

[6] In the special case $m = 2$, (1.4) describes the standard conifold and the dual theory is IR free.

multiplets of $\mathcal{N}=2$ supersymmetry. More generally we compute the exact $\mathcal{N}=1$ F-terms in the low-energy effective action on both sides of the proposed duality and show that they agree precisely. Note that, for the case $m=2$ of the ordinary conifold, we have $p=0$ and the interacting sector has no massless fields. In this special case, we therefore have much less evidence in favour of our conjecture.

The supergravity analysis of the β-deformed theory reviewed above immediately suggests a heuristic interpretation of our proposed correspondence. The full string background dual to the gauge theory with superpotential (1.1) is some complicated warped geometry with strong RR fluxes. The generalised conifold singularity appears when various flux-free cycles vanish. As in the case without flux, the region near the singularity is replaced by an infinite throat with a linear dilaton [49]. Although RR fields are present in the bulk, these fields decay as we move down the throat. In the double-scaling limit, the throat is replaced by a cigar whose length becomes infinite as $N\to\infty$. String states localised near the tip of the cigar decouple from the bulk and have the same world-sheet description as in the case without flux. As in the β-deformed case, many details of the UV theory are lost when the bulk region decouples. This is even more striking in the present case where the dual field theory is non-renormalisable and has an explicit UV cut-off which is held fixed in the limit. In the decoupled sector, all dependence on the cut-off is absorbed onto the two independent parameters of double-scaled LST. We believe that this type of behaviour, which is surely impossible at finite N, is an interesting new aspect of large-N field theory which should have other applications.

The results presented in this paper have some overlap with earlier work. The idea that the standard large-N expansion breaks down near the critical points in the moduli/parameter space of four-dimensional gauge theories has been emphasized in a series of papers by Ferrari [40 − 42], which also contain proposals for double-scaling limits. In particular, one of the limits considered in [42] is closely related to the $m=2$ case of our analysis. Large-N limits leading to dual string backgrounds without RR flux have also been discussed in [33] (see also [19]). The field theory and the specific vacuum studied in this reference also coincide with the $m=2$ case discussed below. However, the specific limit considered in [33] is quite different from the double-scaling limit considered here (this is discussed further in section 6.2). The four-dimensional interpretation of double-scaling in the Dijkgraaf-Vafa matrix model has also been discussed before in [19, 42].

The rest of the paper is organised as follows. Section 2 provides several preliminaries for the analysis of the model (1.1). In particular, we discuss the vacuum structure of the theory and its classification in terms of order parameters. This is mainly a review of parts of [25]. We also review the F-term effective action of [21]. In section 3, we present an analysis of the spectrum and interactions of the model based on large-N counting. We also review basic features of the dual string theory background of [32]. Section 4 identifies the critical point in parameter space and exhibits the breakdown of the $1/N$ expansion. Double-scaling limit is formulated in section 5 and our calculations of the F-terms in this limit are described. A more detailed description of the calculations is given in the appendix. Our proposal for a duality between the model (1.1) and the double-scaled LST of the generalised

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conifold is formulated in section 6. A more detailed review of the β-deformed theory and its double-scaling limit is provided in section 7. Finally, a discussion of our results is given in section 8.

2. The model

In this paper we will mainly be concerned with gauge theories in a phase where confinement is partial in the sense that an abelian subgroup $\hat{G}$ of the original gauge group G remains unconfined and the low-energy theory contains massless gauge fields.[7] We begin by reviewing a family of models which exhibits this behaviour in generic vacuum states. As in section 1 above, we will consider an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $G=\mathrm{U}(N)$. The matter content of the theory is an $\mathcal{N}=1$ vector multiplet, with chiral field strength W_α, and a single adjoint chiral multiplet Φ. The theory has a classical superpotential of the form,

$$W=\varepsilon\mathrm{Tr}_N\left[\frac{\Phi^{m+1}}{m+1}-\sum_{l=1}^{m}g_l\frac{\Phi^l}{l}\right]. \tag{2.1}$$

One obvious objection to this model is that it is non-renormalisable and requires embedding in some larger theory with a sensible UV definition. For our purposes it will be sufficient to introduce an explicit UV cut-off at some scale M_{UV} larger than the other mass scales in the problem.

The classical vacua of the theory are determined by the stationary points of the superpotential (2.1). For generic values of the couplings, we find m stationary points at the zeros of,

$$W'(x)=\prod_{l=1}^{m}(x-x_l)\,. \tag{2.2}$$

The classical vacua are configurations where each of the N eigenvalues of Φ takes one of the m values, $\{x_l\}$, for $l=1,2,\ldots,m$. Thus vacua correspond to partitions of N where $N_l\geq 0$ eigenvalues take the value x_l with $N_1+N_2+\cdots N_m=N$.

Provided $N_l\geq 2$ for all l, the classical low-energy gauge group in such a vacuum is,

$$\hat{G}_{cl}=\prod_{l=1}^{m}\mathrm{U}(N_l)\simeq\prod_{l=1}^{m}\mathrm{U}(1)_l\times\mathrm{SU}(N_l)\,. \tag{2.3}$$

The massless modes in this vacuum comprise an $\mathcal{N}=1$ vector multiplet of $\hat{G}_{cl}$. The classical spectrum also includes massive states which carry electric charges under the unbroken U(1) factors in the low-energy gauge group. In particular there are both vector and chiral multiplets transforming in the $(\mathbf{N}_r,\bar{\mathbf{N}}_s)$ of $\mathrm{SU}(N_r)\times\mathrm{SU}(N_s)$ with charges $(+1,-1)$ under $\mathrm{U}(1)_r\times\mathrm{U}(1)_s$. There are also massive states transforming in the adjoint representation

[7] More precisely, we consider theories with gauge group G which have an unconfined gauge symmetry, $\hat{G}$, at low energies. At least in one regime of parameters (the weak-coupling regime discussed below) it is clear that $\hat{G}$ is a subgroup of the original gauge group G. More generally, $\hat{G}$ and G might not be related in a simple way.

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of each non-abelian factor which are neutral under the unbroken U(1)s. The masses of all these states are set by the dimensionful parameters in the superpotential (2.1). We will denote the lightest such mass-scale M. Finally, the classical theory also includes 't Hooft-Polyakov monopoles which are magnetically charged under pairs of U(1) factors in $\hat{G}_{cl}$. The masses of these states are of order $\tilde{M} = M/g_0^2$ where g_0 is the bare gauge coupling.

2.1 Weak-coupling analysis

Quantum corrections drastically modify the properties of the classical vacuum state described above. The UV gauge theory is asymptotically free and the gauge coupling runs with energy scale. The resulting dynamics is easiest to analyse in a regime where the running coupling $g^2(\mu)$ is small at the scale $\mu = M$ set by the parameters in the superpotential. Following previous authors, we will refer to this as the weak-coupling regime.[8] In terms of the RG invariant scale Λ associated with the running of the U(N) gauge coupling in the UV theory, the weak-coupling regime corresponds to $\Lambda \ll M$. In this regime, the low-energy theory at scales far below M is $\mathcal{N} = 1$ SUSY Yang-Mills with gauge group $\hat{G}_{cl} = \prod_{l=1}^{m} \mathrm{U}(1)_l \times \mathrm{SU}(N_l)$ and we can draw on the standard facts about the dynamics of this theory. In particular, each non-abelian factor in $\hat{G}_{cl}$ is asymptotically free and runs to strong coupling in the IR below the scale,

$$\Lambda_{(l)} \sim M \exp\left(-\frac{8\pi^2}{3N_l g^2(M)}\right). \tag{2.4}$$

The resulting strong coupling dynamics confines all states charged under the non-abelian factors of $\hat{G}_{cl}$ and generates a mass gap of order $\Lambda_{(l)}$ in the l'th factor. The unconfined low-energy gauge-group is $\hat{G} = \mathrm{U}(1)^m = \prod_{l=1}^{m} \mathrm{U}(1)_l$ and the only remaining massless states are m abelian multiplets of $\mathcal{N} = 1$ supersymmetry.

Another effect of the strong-coupling dynamics is to produce non-zero gluino condensates in each non-abelian factor of $\hat{G}_{cl}$. If we denote as $W_{\alpha l}$, the chiral field strength of the SU(n_l) vector multiplet in the low-energy theory, we can define a corresponding low-energy glueball superfield $S_l = -(1/32\pi^2)\langle \mathrm{Tr}_{N_l}(W_{\alpha l} W_l^{\alpha})\rangle$ in each factor. Non-perturbative effects generate a superpotential of the form [34, 25],

$$W_{\mathrm{eff}}(S_1, \ldots, S_m) = \sum_{l=1}^{m} N_l \left(S_l \log\left(\frac{\Lambda_{(l)}^3}{S_l}\right) + S_l\right) + 2\pi i \sum_{l=1}^{m} b_l S_l\,, \tag{2.5}$$

where b_l are integers to be discussed below. Note that this simple additive formula for the superpotential is only valid in the weak-coupling regime $\Lambda \ll M$ where the dynamics of the different non-abelian factors in $\hat{G}_{cl}$ are effectively decoupled. Assuming that the low-energy dynamics is correctly described by an effective action for the fields S_l, we can find the vacuum states of the theory by minimizing W_{eff} to obtain,

$$\langle S_l \rangle = \Lambda_{(l)}^3 \exp\left(\frac{2\pi i}{N_l} b_l\right). \tag{2.6}$$

[8] This is a misnomer because, of course, the coupling is only really weak in the UV.

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The phase of each gluino condensate depends on an integer b_l defined modulo N_l which labels inequivalent supersymmetric vacua. Thus the quantum theory has a total of $N_1 N_2 \dots N_l$ distinct SUSY vacua corresponding to the single classical vacuum we started with.

As first discussed by Cachazo, Seiberg and Witten (CSW) [25], the physics of the partially confining phase depends sensitively on the integers b_l appearing in the phase of the gluino condensate (2.6). In the more familiar case of theories with a mass gap, the possible confining phases were classified by 't Hooft [35] in terms of the electric and magnetic quantum numbers of states which condense in the vacuum. Conventional confinement is associated with the condensation of magnetic degrees of freedom, while various oblique confining phases occur when dyonic states condense. In pure $\mathcal{N}=1$ SUSY Yang-Mills with gauge group SU(n) the corresponding integer $b = 0, 1, 2, \dots, n-1$, appearing in the phase of the gluino condensate, determines which kind of confinement is realised in each vacuum. Precisely one value of b is associated with ordinary confinement and the remaining values lead to oblique confinement. In this case the n vacua are also related by a spontaneously broken discrete symmetry. As the theory contains no dynamical charges in the fundamental representation, the different phases can only be distinguished by introducing external probes. In the present case with several non-abelian factors, the theory contains states carrying electric and magnetic charge under pairs of unconfined U(1)s. In this case the physics of each vacuum depends non-trivially on differences in the mode of confinement between different factors even in the absence of external probes. As there are no states charged under the central U(1), the physics only depends on the differences of the integers b_l. Following [25] we adopt a convention where $b_1 = 0$.

2.2 Order parameters

For a theory containing only fields in the adjoint representation of the gauge group, the natural order parameter for confinement is the Wilson loop,

$$\mathcal{W}_R = \left\langle \mathrm{Tr}_R P \exp\left(-\oint_C A \right) \right\rangle \tag{2.7}$$

where C is a closed contour and the trace is evaluated in some representation R of the gauge group G. The signal for confinement of external charges tranforming in the representation R of G is that the corresponding Wilson line decays exponentially with the area enclosed by the contour C. For an SU(n) gauge theory in a conventional confining phase, this area law holds when R is the fundamental representation $\mathbf{n}$ and also for tensor products of $r < n$ copies of the fundamental representation $\mathbf{n}^r = \mathbf{n} \otimes \mathbf{n} \otimes \dots \otimes \mathbf{n}$. For $r = n$, this tensor product contains a singlet, the baryon, which is not confined and consequently the area law does not hold. More generally, following CSW [25], it is useful to introduce the *confinement index*, which is an integer t defined as the lowest value of r such that the Wilson loop evaluated in the representation $\mathbf{n}^r$ *does not* exhibit the area law. Because the n-fold product always contains a singlet, the confinement index is naturally defined modulo n.

For the vacua of the U(N) theory considered above, the confinement index is easy to compute in the case where the mode of confinement in each non-abelian factor is the same [25]. This corresponds to the choice $b_1 = b_2 = \ldots = b_m = 0$ for the integers defined in the previous section which label the vacua. The fundamental representation $\mathbf{N}$ of SU(N) transforms as,

$$(\mathbf{N}_1, \mathbf{1}, \ldots, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{N}_2, \ldots, \mathbf{1}) \oplus \ldots \oplus (\mathbf{1}, \mathbf{1}, \ldots, \mathbf{N}_m) \tag{2.8}$$

under $\prod_{l=1}^{m} \mathrm{SU}(N_l) \subset \hat{G}_{cl}$. If we take a tensor product of r copies of this representation a singlet will occur if r is equal to any of the integers N_l. Thus a singlet also occurs for $r = l_1 N_1 + l_2 N_2 + \ldots l_m N_m$ for any integers l_l. The smallest non-zero value of r modulo N which can be obtained in this way is equal to the greatest common divisor of the N_l. Thus we find that the confinement index t is equal to the greatest common divisor of the integers N_l with $l = 1, 2, \ldots, m$.

The above discussion takes into account the possibility of the screening of external charges by pair creation of chromoelectric charges. To extend this analysis to the more general vacuum where the integers b_l are not all equal, we also need to consider magnetic screening. This is discussed in detail in [25, section 2.1] and we will only quote the final result: the confinement index in a general vacuum is equal to the greatest common divisor of the N_l and the b_l. We will give a string theory derivation of this result in section 3.3 below.

So far our discussion of confinement has been restricted to the weak coupling regime $\Lambda \ll M$. However, the behaviour of the Wilson loop can only change if we encounter a phase transition. Thus we can interpret the confinement index t as a universal quantity characterising the partially confining phase. Another such quantity is the rank $r = m$ of the low-energy unconfined gauge group $\hat{G}$. The investigation of [25] showed that, as the parameters of the superpotential are varied it is possible to interpolate smoothly between classical vacua corresponding to different partitions of N. Thus the filling fractions N_l/N characterising a particular classical vacuum are not universal in the sense described above. The same applies to the individual integers b_l which determine the mode of confinement in each non-abelian factor. In contrast CSW found that the confinement index t and the rank r of the unconfined gauge group can only change on passing through a singular point where two branches of the moduli space intersect. Other more subtle order parameters which distinguish different branches were also introduced in [25].

2.3 The F-term effective action

In the weak-coupling regime $\Lambda \ll M$, we saw that the low-energy effective theory contains an $\mathcal{N} = 1$ vector multiplet of the unbroken classical gauge group $\hat{G}_{cl} = \prod_{l=1}^{m} \mathrm{U}(N_l)$. From each factor $\mathrm{U}(N_l) \simeq \mathrm{U}(1)_l \times \mathrm{SU}(N_l)$ in $\hat{G}_{cl}$, we obtain a chiral field-strength superfield $W_{\alpha l}$. Interactions between fields living in different factors are mediated by bifundamental diquarks of mass M. Provided we remain at weak coupling these interactions are effectively suppressed by powers of the small parameter Λ/M. More generally, we can obtain corrections to this limit by integrating out the massive degrees of freedom to obtain an effective action for the vector multiplets $W_{\alpha l}$. In principle, a second step would then be to quantize this action to obtain a complete effective action which also takes into account the strong-coupling gauge dynamics at scales of order Λ.

Although this goal is far too ambitious for generic observables, theories with $\mathcal{N}=1$ supersymmetry contain a special subsector of observables which are constrained by holomorphy in complex parameters and background fields. These observables are determined by the F-terms in effective action. Such terms include the effective superpotential for the glueball superfields $S_l = -(1/32\pi^2)\mathrm{Tr}_{N_l}[W_{\alpha l}W_l^\alpha]$, whose weak-coupling form was given in eq. (2.5) above. In a remarkable series of papers [19−21], Dijkgraaf and Vafa determined the exact F-terms in the effective action for $W_{\alpha l}$ obtained by integrating out the massive degrees of freedom. With some plausible additional assumptions their approach also yields the full F-term effective action including the effects of gauge dynamics. The resulting F-term effective action can be written in terms of the glueball superfields S_l and m additional superfields $w_{\alpha l} = (1/4\pi)\mathrm{Tr}_{N_l}[W_{\alpha l}]$, for $l=1,2,\ldots,m$ which contain as components the massless photons and their $\mathcal{N}=1$ superpartners. The general form of the action is,

$$\mathcal{L}_F = \mathrm{Im}\left[\int d^2\theta\, W_{\mathrm{eff}}\right] = \mathrm{Im}\left[\int d^2\theta\, W_{\mathrm{eff}}^{(0)} + W_{\mathrm{eff}}^{(2)}\right] \tag{2.9}$$

with,

$$\begin{aligned} W_{\mathrm{eff}}^{(0)} &= \sum_l N_l \frac{\partial\mathcal{F}}{\partial S_l} + 2\pi i\tau_0 \sum_l S_l + 2\pi i \sum_l b_l S_l \\ W_{\mathrm{eff}}^{(2)} &= \frac{1}{2}\sum_{k,l} \frac{\partial^2\mathcal{F}}{\partial S_k \partial S_l}\, w_{\alpha k} w_l^\alpha\,. \end{aligned} \tag{2.10}$$

To avoid any confusion we will refer to W_{eff} as the total superpotential and $W_{\mathrm{eff}}^{(0)}$ on its own as the glueball superpotential. The function $\mathcal{F}(S_1, S_2, \ldots, S_m)$ is known as the prepotential.

The exact F-terms are governed by the corresponding spectral curve Σ,

$$y^2 = W'(x)^2 - f(x) = \varepsilon^2 \prod_{l=1}^m \left(x - x_+^{(l)}\right)\left(x - x_-^{(l)}\right), \tag{2.11}$$

where $f(x) = \sum_{l=1}^m \kappa_l x^{l-1}$ depends on m parameters κ_l which will be determined below. The curve is a double-cover of the complex x plane branched over the $2m$ points $x_\pm^{(l)}$ for $l=1,2,\ldots,m$, with a marked point at infinity on both sheets. The two sheets of the cover are joined along m branch cuts C_l. We chose the cuts so that C_l runs from $x_+^{(l)}$ to $x_-^{(l)}$ for $l=1,2,\ldots m$. We also introduce the following basis of one-cycles A_k and B_k with intersections,

$$A_k \circ A_l = B_k \circ B_l = 0 \qquad A_k \circ B_l = \delta_{kl}\,. \tag{2.12}$$

The cycle A_k surrounds the cut C_k. The (non-compact) dual cycle B_k runs from the point at infinity on the lower sheet to the same point on the upper sheet, passing from the lower to the upper sheet by traversing the cut C_k. The resulting non-compact Riemann surface

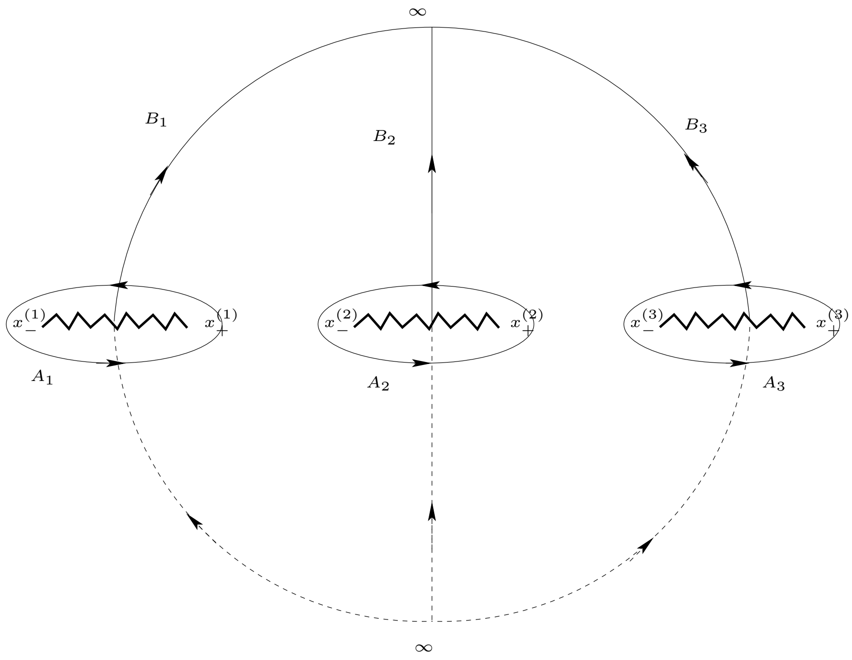

Figure 1: The spectral curve Σ for $m = 3$.

has genus $m-1$. The surface Σ and its one-cycles are shown in figure 1 for the case $m = 3$. The solution is then determined by the periods of the differential ydx according to,

$$S_l = \oint_{A_l} ydx \qquad S_l^D = \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial S_l} = \oint_{B_l} ydx\,. \tag{2.13}$$

In particular, the first equation in (2.13) effectively determines the m parameters κ_l in terms of the S_l. The second equation in (2.13) can then be used to express $\mathcal{F}$ as a function of the glueball superfields. This in turn determines the full F-term action (2.9), (2.10). The abelian gauge couplings appearing in (2.10) are determined by the matrix of second partial derivatives of $\mathcal{F}$ which can be thought of as a generalised period matrix for the non-compact Riemann surface Σ.

As we are interested in moving away from the weak coupling limit, the definitions of the chiral superfields S_l and $w_{\alpha l}$ used above are not really satisfactory because they are only invariant under the classically unbroken subgroup $\hat{G}_{cl}$, and not under the full U(N) gauge symmetry of the theory. A very elegant gauge-invariant definition of these observables, which reduces to the definition given above at weak-coupling, was suggested in [24]. The definition involves the closed contours, A_l, in the complex x-plane introduced above. It is easy to check that each contour A_l surrounds exactly one critical point, x_l, of the classical superpotential $W(x)$. We can then define,

$$\begin{aligned} S_l &= -\frac{1}{2\pi i}\oint_{A_l} dx\,\frac{1}{32\pi^2}\mathrm{Tr}_N\left[\frac{W_\alpha W^\alpha}{x-\Phi}\right] \\ w_{\alpha l} &= \frac{1}{2\pi i}\oint_{A_l} dx\,\frac{1}{4\pi}\mathrm{Tr}_N\left[\frac{W_\alpha}{x-\Phi}\right]. \end{aligned} \tag{2.14}$$

3. The large-N limit

In this section we will consider the large-N behaviour of the theory in its partially confining phase. We begin by briefly reviewing the standard discussion of SU(N) Yang-Mills theory with and without quarks (see for example [36]). In subsequent subsections we adapt these standard counting arguments to the case of partial confinement, discuss the F-term effective action at large-N and finally discuss the dual string theory.

3.1 A review of large-N counting

We will begin by reviewing the standard large-N analysis of pure Yang-Mills theory with gauge group SU(N). A similar discussion applies to any (totally) confining theory with all fields in the adjoint representation of the gauge group. To describe the large-N limit, it is convenient to rescale the fields so that the bare gauge coupling g_0^2 appears only via an overall prefactor of $1/g_0^2$ in front of the action. The 't Hooft limit then corresponds to taking $N \to \infty$ and $g_0^2 \to 0$ with the combination $g_0^2 N$ held fixed. In terms of renormalisation group invariant quantities, this corresponds to a limit where $N \to \infty$ with the dynamical scale Λ held fixed. Thus the mass gap of the theory remains constant in the limit.

The spectrum of pure Yang-Mills theory consists of colour-singlet glueballs with masses of order Λ. At large-N, the corresponding single-particle states are created by gauge-invariant single trace operators acting on the vacuum. With the conventions described above, where N appears as an overall prefactor in front of the action, single-trace operators create single-particle states with amplitude of order one. For example scalar glueballs are created with amplitude of order one by the operator $\hat{O} = \mathrm{Tr}_N[F_{\mu\nu}F^{\mu\nu}]$. Connected correlation functions of such operators obey the scaling rule,

$$\langle \hat{O}(x_1)\hat{O}(x_2)\ldots\ldots\hat{O}(x_L)\rangle \sim N^{2-L}\,. \tag{3.1}$$

This means that glueball masses, which are determined by the exponential decay of two-point functions, remain fixed in the large-N limit, while decay widths corresponding to three-point functions go like $1/N$. On shell $2 \to 2$ scattering amplitudes are extracted from four-point functions which scale like $1/N^2$. As $N \to \infty$, we therefore expect a free theory with an infinite tower of stable glueballs. This is exactly the behaviour expected from a closed string theory with string coupling $g_s \sim 1/N$.

Although the supersymmetric models considered in this paper contain only adjoint fields, it will be useful to extend our discussion of pure SU(N) Yang-Mills theory by adding massive quarks in the fundamental representation. In this case the glueball spectrum described above will be accompanied by massive mesons and baryons. Standard large-N counting indicate that mesons have masses of order one and L-point interactions which scale like $N^{1-L/2}$. Thus mesons, like glueballs, become free in the large-N limit. In a dual string theory description with string coupling $g_s \sim 1/N$ mesons are naturally identified as excited states of an open string. Mixing between mesons and glueballs is also suppressed in the large-N limit. Baryons have very different properties at large-N [37]. Their masses grow linearly with N reflecting the presence of N constituent quarks, while their shape and size remain fixed at large-N. A key feature which will play an important role in the following is that the interactions of baryons, both with mesons and with glueballs, are *not* suppressed in the large-N limit. In particular the interactions of glueballs with baryons are governed by matrix elements of single trace operators such as $\hat{O} = \mathrm{Tr}_N[F_{\mu\nu}F^{\mu\nu}]$ between one-baryon states. Applying standard counting rules we find,

$$\langle B|\hat{O}(x_1)\hat{O}(x_2)\ldots\ldots\hat{O}(x_L)|B'\rangle \sim N^{1-L}\,. \tag{3.2}$$

Thus the three-point coupling between two baryons and a single glueball corresponds to the case $L = 1$ and therefore scales like N^0. A similar coupling for two baryons and a single meson grows like $N^{\frac{1}{2}}$. All of these features are easily understood from the perspective of the dual string theory where glueballs and mesons are identified with the excitations of closed and open strings respectively. In this context baryons should be thought of as D-particles on which open strings can end [38]. The leading contribution to the matrix element (3.2) corresponds to a disc amplitude coupling the D-brane to L closed string states which scales as $g_s^{L-1} \sim N^{1-L}$.

3.2 Partially confining theories at large-N

How does the above story change when we have a theory in the partially confining phase? The obvious difference is the presence in the spectrum of massless gauge bosons of the unconfined gauge group $\hat{G} = \mathrm{U}(1)^m$. All the large-$N$ limits we will consider in this paper are ones where the rank m of the low-energy gauge group is held fixed. Heuristically, the presence of the low-energy gauge symmetry is only significant if there are states in the spectrum which carry the corresponding charge. In the absence of light charged states the gauge bosons decouple in the IR. In the following we will see that the spectrum does contain states which are electrically and/or magnetically charged under $\hat{G}$. However, in generic vacua, these states become very massive at large N (somewhat like the baryons of large-N QCD described in the previous subsection) and we will argue that the remaining spectrum of $\hat{G}$-neutral states resembles that of a standard large-N gauge theory.

We will now describe the large-N limit of the partially confining phase in more detail. As before we take the $N \to \infty$ with $g_0^2 N$ fixed, which corresponds to holding the dynamical scale Λ fixed. Again, it is convenient to normalize the bare action so that a power of $1/g_0^2$ appears as a prefactor for the vector multiplet kinetic terms. We also adopt conventions where the same prefactor appears in front of the kinetic term for the chiral multiplet. Finally, to ensure that the remaining terms in the action also scale linearly with N, we also take the parameter ε appearing in the superpotential (2.1) to infinity with $\tilde{\varepsilon} = \varepsilon/N$ held fixed. The remaining superpotential couplings g_l are held fixed. The only remaining choice is the scaling of the filling fractions, N_l/N, which characterise the vacuum. We will restrict our attention to the simplest possibility, where the filling fractions approach constant values as $N \to \infty$.

To analyse the resulting spectrum, it is simplest to start our discussion by considering the weak-coupling regime $\Lambda \ll M$. In this case the effective theory at energy scales below M is $\mathcal{N} = 1$ SUSY Yang-Mills with gauge group $\hat{G}_{cl} = \prod_{l=1}^{m} \mathrm{U}(1)_l \times \mathrm{SU}(N_l)$. The large-$N$ limit defined above involves scaling each N_l like N and also holds the corresponding dynamical scales $\Lambda_{(l)}$ fixed. Thus our chosen limit reduces at low energies to the standard 't Hooft limit of the $\mathcal{N}=1$ SUSY Yang-Mills theory in each non-abelian factor of $\hat{G}_{cl}$. The theory in the $\mathrm{SU}(N_l)$ factor has a mass gap of order $\Lambda_{(l)} \sim N^0$ and the spectrum includes an infinite tower of weakly-interacting stable glueballs with masses which remain fixed as $N \to \infty$. As before, the spectrum also includes the m massless photons of the unconfined gauge-group $\hat{G} = \mathrm{U}(1)^m$.

The remaining degrees of freedom of the full $\mathrm{U}(N)$ gauge theory acquired masses of order M in the classical theory. These include states Q_{rs} which transform in the $(\mathbf{N}_r, \mathbf{N}_s)$ representation of $\mathrm{SU}(N_r) \times \mathrm{SU}(N_s) \subset \hat{G}_{cl}$ and carry electric charges $(+1, -1)$ under the unconfined abelian subgroup $\mathrm{U}(1)_r \times \mathrm{U}(1)_s \subset \hat{G}$. From the point of view of the low-energy gauge-group factors $\mathrm{SU}(N_r)$ and $\mathrm{SU}(N_s)$ these states are heavy diquarks with large bare masses of order $M \gg \Lambda$. The abelian subgroups $\mathrm{U}(1)_r$ and $\mathrm{U}(1)_s$ correspond to a gauged version of the corresponding baryon number symmetries. At low-energies, the diquarks will be confined to form singlets of $\mathrm{SU}(N_r) \times \mathrm{SU}(N_s)$, including dimesons $Q_{rs}\bar{Q}_{rs}$ and dibaryons $Q_{rs}Q_{rs}\ldots Q_{rs}$ where the minimum number, N_{rs}, of diquarks needed to form a colour-singlet is the lowest common multiple of the integers N_r and N_s.

In the weak-coupling limit, the properties of the dimesons and dibaryons can be understood using the standard large-N analysis of QCD with massive quarks reviewed above. In particular, as the dynamics of different non-abelian factors are effectively decoupled, their contributions to dibaryon and dimeson masses are additive. Thus we deduce that these states have masses of order N and N^0 respectively. The three-point coupling of any two dibaryons charged under a $\mathrm{U}(1)_l \subset \hat{G}$ with a single glueball of $\mathrm{SU}(N_l)$ is of order N^0. The only subtlety that arises concerns dimeson interactions. In particular, a three-point self-coupling for dimesons of $\mathrm{SU}(N_r) \times \mathrm{SU}(N_s)$ gains a suppression factor $\sim 1/\sqrt{N}$ from each factor and thus scales like $1/N$ just like the three-point glueball self-couplings. Similar counting leads to the conclusion that the large-N properties of dimesons are essentially the same as those of glueballs.[9]

The classical theory also contained magnetic monopoles of mass $\tilde{M} \sim M/g_0^2$. If we focus on the vacuum with $b_1 = b_2 = \ldots = b_m = 0$ where confinement is due to magnetic condensation in each factor, these states will remain unconfined. However their masses will be renormalised in the quantum theory. We can estimate the renormalised mass by replacing the bare coupling g_0^2 by the running coupling $g^2(\mu)$ evaluated at the scale $\mu = M$ of classical symmetry breaking. Further renormalisation can occur due to strong coupling effects in the IR which produce additive correction of order $\Lambda_{(l)}$. As $g^2(M)$ goes like $1/N$ in the large-N limit defined above, we deduce that the masses of magnetically charges states grow linearly with N. As discussed above, the masses of electrically charged dibaryons also grow with N as $N \to \infty$. There are other interesting parallels between the large-N behaviour of magnetic monopoles and of dibaryons [37]. Monopoles are classical field configurations whose size and shape are independent of g_0^2 and thus of N. Interactions between elementary quanta and these classical solitons also remain unsuppressed as $N \to$

[9]This reflects the fact that the theory contains only adjoint fields, which means that (in the absence of D-branes) the dual string theory contains only closed strings. In particular both glueballs and dimesons correspond to closed string excitations.

∞. In the following we will see that magnetic monopoles and dibaryons appear on an equal footing in the dual string theory as wrapped/stretched D-branes. We will also come across dyonic particles corresponding to bound-states of these objects, as well as interesting low-energy electric-magnetic dualities which permute all of these states. For ease of reference, we will sometimes refer to the $\hat{G}$-charged states collectively as dibaryons.

So far our discussion of the large-N spectrum has been limited to the weak-coupling limit $\Lambda \ll M$. Moving away from this limit, we can no longer rely on the low-energy description in terms of a $\hat{G}_{cl} = \prod_{l=1}^{m} \mathrm{U}(N_l)$ gauge theory. In general states can only be distinguished by their quantum numbers under the true low-energy gauge group $\hat{G} = \mathrm{U}(1)^m$. Thus, as we increase the ratio Λ/M the $\hat{G}$-neutral glueballs and dimesons described above will mix and it no longer makes sense to distinguish between them. On the other hand the electric and magnetic dibaryons are still distinguished by their charges under $\hat{G}$. In the following we will see that the following qualitative aspects of the large-N spectrum described above persist beyond the weak-coupling coupling limit,

1. States neutral under $\hat{G}$ have masses of order N^0 and interactions (with other neutral states) suppressed by powers of $1/N$.

2. States carrying electric and/or magnetic charges under $\hat{G}$ have masses which grow linearly (or faster) with N.

3. Interactions between $\hat{G}$-charged and $\hat{G}$-neutral states remain unsuppressed at large-N.

3.3 F-terms and the large-N limit

In this subsection we will consider the exact F-term effective action (2.9), (2.10) of Dijkgraaf and Vafa in the context of the large-N limit. The effective action is written in terms of chiral superfields S_l and $w_{\alpha l}$ which are defined as gauge-invariant single-trace operators in (2.14). It will also be convenient to define component fields for each of these superfields,

$$\begin{aligned} S_l &= s_l + \theta_\alpha \chi_l^\alpha + \cdots \\ w_{\alpha l} &= \lambda_{\alpha l} + \theta_\beta f_{\alpha l}^\beta + \cdots . \end{aligned} \tag{3.3}$$

The component fields, s_l and f_l are bosonic single trace operators while χ_l and λ_l are fermionic single trace operators. In the large-N limit, these operators should create bosonic and fermionic colour-singlet single particle states respectively. It is instructive to consider the interaction vertices for these fields contained in the F-term effective action. Expanding (2.9) in components we find terms like,

$$\mathcal{L}_F^{(2)} = \int d^2\theta \, W_{\text{eff}}^{(2)} \supset V_{ij}^{(2)} f_{\alpha\beta}^i f^{\alpha\beta j} + V_{ijk}^{(3)} \chi_\alpha^i f^{\alpha\beta j} \lambda_\beta^k + V_{ijkl}^{(4)} \chi_\alpha^i \chi^{\alpha j} \lambda_\beta^k \lambda^{\beta l} , \tag{3.4}$$

where,

$$V_{i_1 i_2 \ldots i_L}^{(L)} = \left\langle \frac{\partial^L \mathcal{F}}{\partial S_{i_1} \partial S_{i_2} \ldots \partial S_{i_L}} \right\rangle \tag{3.5}$$

for $L = 2, 3, 4$. We will also consider the 2-point vertex coming from the glueball superpotential $W^{(0)}_{\rm eff}$,

$$\mathcal{L}^{(0)}_F = \int d^2\theta\, W^{(0)}_{\rm eff} \supset H^{(2)}_{ij}\chi^i_\alpha\chi^{\alpha j}\,, \tag{3.6}$$

where

$$H^{(2)}_{ij} = \left\langle \frac{\partial^2 W^{(0)}_{\rm eff}}{\partial S_i \partial S_j} \right\rangle \tag{3.7}$$

The matrix $H^{(2)}_{ij}$ therefore effectively determines the masses of the chiral multiplets S_l.

In the large-N limit, the leading contribution to these vertices should come from planar Feynman diagrams. By the standard counting arguments reviewed in section 3.1, the planar contribution to the L-point vertex $V^{(L)}$ should scale like N^{2-L}. Similarly the two-point vertex $H^{(2)}$ scales like N^0. For these F-terms, the form of the perturbation series is also constrained by holomorphy and dimensional analysis [21, 24]. To simplify the analysis, we will focus on a particular submanifold in parameter space defined by $g_1 = \mathcal{U}$, $g_l = 0$ for $l > 1$, where the superpotential (2.1) preserves a $\mathbb{Z}_m$ symmetry. In this case, the series of planar contributions to the three-point coupling has the general form,

$$V^{(3)}_{ijk} \sim \frac{1}{\varepsilon\mathcal{U}^{\frac{m+1}{m}}} \sum_{r=0}^{\infty} C^{(ijk)}_{i_1 i_2 \ldots i_r} \hat{S}_{i_1}\hat{S}_{i_2}\ldots\hat{S}_{i_r} \tag{3.8}$$

where $\hat{S}_l = (S_l/\varepsilon\mathcal{U}^{\frac{m+1}{m}})$. The r'th term in the series (3.8) comes from a sum of planar diagrams with $r+2$ loops. To keep track of the N dependence recall that $\varepsilon \sim N$, $\mathcal{U} \sim N^0$ and that the expectation value of a single-trace operator S_l scales like N. Thus we confirm that the planar contribution to $V^{(3)}_{ijk}$ scales like $1/N$ as expected for a vertex coupling three single-trace operators.

The exact formulae (2.10), (2.11), (2.13) of Dijkgraaf and Vafa effectively determine the coefficients $C^{(ijk)}$ in the expansion (3.8). From our point of view these results have two remarkable features. The first is simply that, for these special F-term quantities, the full sum over planar diagrams can be performed. This is the first example where this is possible in an interacting four-dimensional field theory. In particular, such a resummation for generic observables in the same theory is still far out of reach. The simplification can be understood as a cancellation of four-dimensional kinematic factors in each planar diagram reducing the calculation to a sum over the planar Feynman diagrams of an associated zero-dimensional matrix model. The second feature is that the results are given exactly by the planar contribution even for finite N. Thus there are no $1/N$ corrections to the leading order result. This feature relies heavily on holomorphy and is therefore also special to the F-term observables. The absence of corrections at any order in the $1/N$ expansion[10] means that the Dijkgraaf-Vafa results hold equally well at finite N. In this paper we are mainly interested in the large-N limit and will mainly exploit Dijkgraaf and Vafa's results as providing examples where the leading order can be computed exactly.

[10] This fact is usually emphasized by introducing a new variable $\hat{N}$ for the dimension of the matrices of the associated matrix model which is different from N. The sum over zero-dimensional planar diagrams is then generated by considering an $\hat{N} \to \infty$ limit with N fixed.

3.4 Large-N string dual

A large-N string theory dual for the model with generic superpotential (2.1) has been proposed by Cachazo, Intriligator and Vafa [32]. In their construction the U(N) gauge theory in question is realised by wrapping N D5 branes on topologically non-trivial two-spheres in a certain Calabi-Yau three-fold. In particular, the second homology group of the three-fold contains a basis of m non-intersecting two-cycles each homeomorphic to S^2. Labelling these two-spheres as S^2_l for $l = 1, 2, \ldots, m$, the classical vacuum with low energy gauge group $\hat{G}_{cl} = \prod_{l=1}^{m} \mathrm{U}(N_l)$ is realised by wrapping N_l D5 branes on S^2_l for each l. The couplings $\{g_l\}$ appearing in the superpotential (2.1) correspond to geometrical parameters controlling the complex structure of the Calabi-Yau manifold. As noted above, the theory described by the superpotential (2.1) is non-renormalisable, and the string theory construction implicitly provides some UV completion of the model. In this case the UV theory is related to the six-dimensional theory living on the five-branes. Unwanted additional degrees of freedom appear in the form of Kaluza-Klein modes of mass $\sim 1/R$ arising from compactification of the six-dimensional theory on an S^2 of radius R. As in other examples [5], R needs to be *large* in string units ensure a large hierarchy between the UV scale $M_{UV} \sim R^{-1}$ and the scale Λ of four-dimensional gauge dynamics.[11]

The large-N duality of [32], corresponds to a geometric transition in which the D5 branes disappear and are replaced by flux. More precisely, the two-spheres S^2_l on which the branes are wrapped are blown down to zero size, while m non-intersecting three-spheres S^3_l, are blown up to finite size. The resulting Calabi-Yau geometry is defined by the complex equation,

$$u^2 + v^2 + w^2 + W'(x)^2 - f(x) = 0\,. \tag{3.9}$$

Here $f(x)$ is the same function appearing in the definition (2.11) of the spectral curve Σ. Indeed the three-fold (3.9) can be thought of as fibration of an a_1 singularity over the Riemann surface Σ. In this context, the homology three-sphere S^3_l corresponds to the product of the unique compact two-cycle of the a_1 singularity, with the one-cycle A_l on Σ defined above. The resulting closed string background includes N_l units of Ramond-Ramond three-form flux through this cycle. In the absence of flux, the low-energy theory contains an abelian vector multiplet of $\mathcal{N} = 2$ SUSY associated with each three-cycle S^3_l. The scalar component of each vector multiplet is the holomorphic volume of the corresponding three-cycle. Including RR flux through the basis three-cycles introduces a superpotential which breaks the $\mathcal{N} = 2$ supersymmetry down to an $\mathcal{N} = 1$ subalgebra. As in the field theory, the remaining massless fields comprise m abelian vector multiplets of $\mathcal{N} = 1$ SUSY.

In principle, the geometric transition described above provides a dual closed string theory formulation of the gauge theory in its partially confining phase. Unfortunately, the resulting geometry is complicated because of the presence of strong background RR fields due to the flux through the three-spheres. The corresponding solutions of the supergravity field equations are unknown and would probably be irrelevant anyway due to the presence

[11]For the theory of N D5 branes wrapped on a two-sphere of radius R we find that $\Lambda/M_{UV} \sim \exp(-4\pi^2/Ng^2_{UV})$ where the UV gauge coupling g^2_{UV} is proportional to α'/R^2

of large α' corrections. For this reason, applications of the geometric transition have mainly been confined to determining F-terms which are insensitive to the presence of background flux.

Despite the problems described above, we may still make a few qualitative remarks about the dual string theory and compare it with our expectations of the large-N field theory. The first point, emphasized in [39], is that the large-N duality implied by the geometric transition is of a standard form. In particular, the dual string coupling g_s scales like $1/N$ as anticipated by 't Hooft, while the string mass scale $M_s = 1/\sqrt{\alpha'}$ is identified with the dynamical scale Λ of the gauge theory. Thus, as in other confining gauge theories, we should identify the excited states of the dual string with the tower of stable glueballs which is expected at large N.

Other features of the large-N gauge theory discussed above are also visible in the string theory dual. In particular, we can identify states which are electrically and/or magnetically charged with respect to the unconfined U(1)'s in the low-energy gauge group. These states correspond to wrapped D3 branes around some of the non-contractible three-cycles. If a three-cycle has n units of threeform flux, a wrapped D3 brane by itself is not consistent with charge conservation [38]. Instead, well-known selection rules demand that n fundamental strings must end on the wrapped brane. The resulting object corresponds to a baryon vertex in the dual field theory. The basis cycle S^3_l introduced above carries N_l units of threeform flux. A D3 brane wrapped around S_l, corresponds to the baryon vertex of the $\mathrm{SU}(N_l)$ factor in the low-energy gauge group $\hat{G}_{\mathrm{cl}}$. However a dibaryon of $\mathrm{SU}(N_r)\times\mathrm{SU}(N_s)$ can be realised by wrapping N_{rs}/N_r D3 branes on S^3_r and N_{rs}/N_s anti-D3 branes on S^3_s where, as above, N_{rs} is the lowest common multiple of N_r and N_s. To accomodate the selection rule described above, we must also join the two sets of wrapped branes with N_{rs} fundamental strings. The mass of the resulting dibaryon then scales like N_{rs} as expected from field theory. The large-N scaling of glueball-dibaryon interactions obtained by field theory arguments in the previous section also follows immediately we identify the dibaryon as a D-brane.

In addition to the basis cycles S^3_l there are also additional compact cycles in the geometry. These include linking cycles with intersection numbers $(+1,-1)$ with respect to pairs of the basis cycles S^3_l. A D3 brane wrapped on a cycle $S^3_{r,s}$ linking S^3_r and S^3_s corresponds to a monopole carrying magnetic charges $(+1,-1)$ under $\mathrm{U}(1)_r\times\mathrm{U}(1)_s\subset\hat{G}$. As before such a wrapped brane is only allowed to exist in isolation if there is no flux through the linking cycle. As shown in [25], the number of units of flux through each linking cycle is determined by the integers b_l which measure the differences between the modes of confinement in each vacuum. In particular, the number of units of flux through $S^3_{r,s}$ is equal to $b_r - b_s$. Thus a wrapped brane is allowed only if $b_r = b_s$ which is true only in a vacuum where the modes of confinement in the non-abelian factors $\mathrm{SU}(n_r)$ and $\mathrm{SU}(n_s)$ are the same. This is simply the condition that the corresponding magnetic (or dyonic) state remains unconfined in the dual field theory. The masses of the wrapped D-branes scale like $1/g_s \sim N$ giving the expected large-N scaling for magnetic monopoles. The fact that dibaryons and magnetic monopoles both correspond to wrapped D-branes also accounts for the similarities between the two types of states noted above.

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Finally it is also straightforward to check the formula of [25] for the confinement index t in a generic vacuum quoted in the previous section. The integer t is the smallest positive integer such that an external charge in the representation of $\mathrm{U}(N)$ formed from the product of t copies of the fundamental representation can be screened. In the string dual such an external charge can be formed from t fundamental strings ending on the boundary (ie at infinity). Screening will occur if we can find a configuration of wrapped branes with finite energy on which these strings can end. To achieve this we wrap k_l D3 branes around each basis cycle S^3_l and m_l D3 branes around the linking cycle $S^3_{l,l+1}$. Here negative values of k_l or m_l correspond to anti-D3 branes. In addition to the t external strings ending on the branes we can also join branes wrapping different cycles with additional fundamental strings. This can be done provided that we can choose the integers k_l and m_l such that,

$$\sum_{l=1}^{m} k_l N_l + m_l b_l = t \qquad \mod N\,. \tag{3.10}$$

The smallest positive integer t satisfying this requirement is the greatest common divisor of the N_l and the b_l. Thus we find the expected value of the confinement index [25].

4. A critical point

So far we have discussed the theory at generic points in a partially confining phase. In these generic vacua, the masses of all states charged under the unconfined gauge group $\hat{G}$ grow linearly (or faster) with N. As $N \to \infty$ these charged states decouple leaving a conventional large-N spectrum of $\hat{G}$-neutral glueballs. In particular, the standard scaling law (3.1) for the correlators of single trace operators holds. Equivalently, glueballs are free at $N = \infty$ and their interactions can be calculated in a $1/N$ expansion. In this section we will consider critical points in the parameter space where this simple picture breaks down.

As we move around in the space of complex parameters the masses of the electrically charged dibaryons and of the magnetic monopoles will change. A natural possibility to consider is that of singular points where one or more of these states becomes massless. In the context of the string dual described in the previous section, the charged states correspond to D3 branes wrapped around non-contractible three-cycles in the geometry, specifically those cycles which do not have RR flux through them. The states in question become massless when the corresponding cycles shrink to zero size. These states are not BPS and we do not know their exact masses. However $\mathcal{N} = 1$ F-terms in these theories can be computed exactly for any values of the parameters and these include the low-energy U(1) couplings. As in theories with $\mathcal{N} = 2$ supersymmetry, the presence of light charged states can be detected via the resulting logarithmic running of these couplings.

Another consequence of light charged states is that the simple picture of the large-N limit decribed above no longer holds. This is particularly clear when we recall that interactions between glueballs and $\hat{G}$-charged states remain unsuppressed in the large-N limit. Noting the result of the previous section that the three-point coupling V of a single glueball to two charged states is of order N^0, we may estimate the contribution of loops of charged states to the interaction vertices of L glueballs. A one-loop contribution to

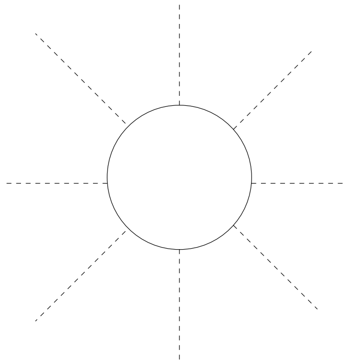

Figure 2: A one-loop contribution to an L-point interaction vertex (for $L = 8$).

the L-point interaction vertex with fermionic charged states of mass $M_B \sim N$ running around the loop is shown in figure 2. Without taking into account the index structure of the vertices or the constraints of supersymmetry we will obtain a rough estimate of this contribution which will suffice to illustrate the main points. We evaluate the loop integral with a UV cut-off $M_{UV} \sim N^0$. The behaviour at large-N is given by,

$$V^L \int_{p^2 < M_{UV}^2} d^4p \, \mathrm{tr} \left(\frac{\gamma_\mu p^\mu + M_B}{p^2 + M_B^2} \right)^L \sim V^L M_B^L \int_0^{M_{UV}^2} \frac{x dx}{(x + M_B^2)^L} \sim M_{UV}^4 \left(\frac{V}{M_B} \right)^L . \quad (4.1)$$

As $M_B \sim N$, this is consistent with the standard large-N counting rule which says that each additional glueball adds one power of $1/N$. However, we can also see that the loop contribution (4.1) will diverge as we approach a critical point where M_B vanishes. The divergence grows worse as we increase L, violating standard large-N scaling. This is our first indication that the conventional $1/N$ expansion breaks down at the critical point.

As mentioned above, the presence of critical points where new massless charged states appear can be detected using the exact F-term action of Dijkgraaf and Vafa. In this paper, we will focus on critical points where the dual string background (3.9) develops a generalised conifold singularity of the form (1.4). In the following, we will relate the violation of large-N scaling described above to the well-known breakdown of closed string perturbation theory at the singularity. In the light of the relationship between the spectral curve Σ and the dual Calabi-Yau, the critical point corresponds to a point where Σ develops a singularity of the form $y^2 = x^m$. Such points have been studied previously by several authors [42 – 45]. The particular case relevant to our present study was considered in [44].

To identify and analyse the critical point, it is useful to proceed via two steps,

1. Off-shell critical points can be identified as points in field space with coordinates where the F-term effective action (2.10) becomes singular. This corresponds to finding special values of the gluino superfields $\{S_1, S_2, \ldots, S_m\}$ where the spectral curve Σ degenerates.

2. Having found the off-shell critical point, the second step is to find a supersymmetric vacuum where the criticality is realised on-shell. In other words we must find a

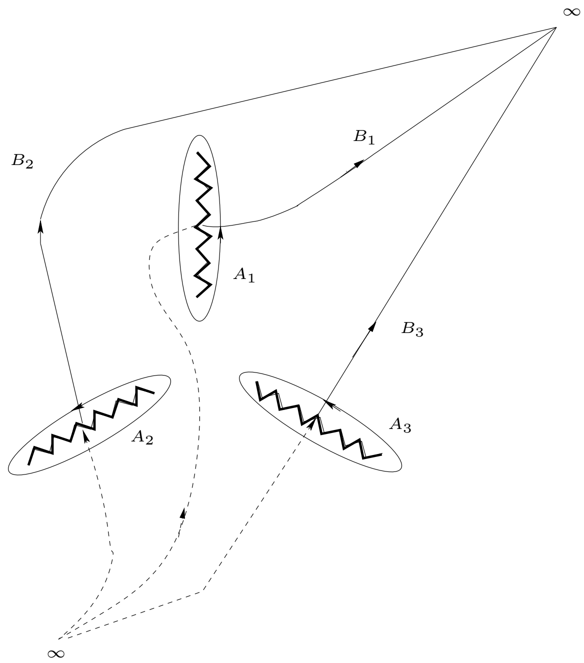

Figure 3: The A and B cycles for $m = 3$.

supersymmetric vacuum where the critical values of the gluino superfields are attained as vacuum expectation values.

To exhibit the off-shell critical point, we will focus on the special case $g_1 = \mathcal{U}$ with $g_l = 0$ for $l > 1$, where the superpotential (2.1) has a $\mathbb{Z}_m$ symmetry. It is also convenient to restrict our attention to a particular submanifold in the field space which respects this symmetry: we set $S_l = S\exp(2\pi i l/m)$, for $l = 1, 2, \ldots, m$. This in turn is equivalent to the choice $\kappa_l = 0$ for all $l > 1$, for the coefficients of the polynomial $f(x)$ in (2.11). Ultimately, these choices will be realised on-shell by stationarising the glueball superpotential.

With these choices the spectral curve becomes,

$$y^2 = \varepsilon^2(x^m - \mathcal{U})^2 - \kappa_1 \tag{4.2}$$

with branch points which lie at,

$$x_{\pm}^{(l)} = \left(\mathcal{U} \pm \frac{\sqrt{\kappa_1}}{\varepsilon}\right)^{\frac{1}{m}} e^{\frac{2\pi i l}{m}} . \tag{4.3}$$

The corresponding Riemann surface Σ is shown for the case $m = 3$ in figure 3.

The critical point of interest occurs for $\sqrt{\kappa_1} = \pm\mathcal{U}\varepsilon$. Choosing the positive sign, we see that the m branch points $x_{-}^{(l)}$, for $l = 1, 2, \ldots, m$, collide at the origin and the curve becomes

$$y^2 = \varepsilon^2 x^m (x^m - 2\mathcal{U}) \tag{4.4}$$

which has the desired singularity of the form $y^2 \sim x^m$ at the origin. Using the first equation in (2.13), we find that the gluino superfields take the values $S_l = S_{cr}\exp(2\pi i l/m)$ with

$$S_{cr} = d_m \varepsilon \mathcal{U}^{\frac{m+1}{m}} \tag{4.5}$$

at the critical point. The overall numerical prefactor is given by,

$$d_m = -\frac{\Gamma(\frac{m+2}{2m})}{\Gamma(\frac{1}{2m})\Gamma(\frac{3}{2}+\frac{1}{2m})}\,. \tag{4.6}$$

The next step is to find SUSY vacua in which the critical value of the glueball superfields is attained as a vacuum expectation value. To do this we must minimise the full effective superpotential,

$$W_{\rm eff}^{(0)} = \sum_{l=1}^{m} N_l \frac{\partial \mathcal{F}}{\partial S_l} + 2\pi i \tau_0 \sum_{l=1}^{m} S_l + 2\pi i \sum_{l=1}^{m} b_l S_l\,. \tag{4.7}$$

As explained above the integers N_l determine the pattern of classical symmetry breaking while the integers b_l correspond to a choice of vacuum in the quantum theory. The problem of minimising the superpotential (4.7) is fully equivalent to that of finding a factorisation of the associated $\mathcal{N}=2$ curve [32] of the form,

$$y^2_{\mathcal{N}=2} = P_N(x)^2 - 4\Lambda^{2N} = H_{N-m}(x)^2 y^2\,, \tag{4.8}$$

where P_N and H_{N-m} are polynomials of degree N and $N-m$ repectively and, as above, y is given by equation (2.11) which defines the $\mathcal{N}=1$ spectral curve Σ. Once a factorisation has been found, the values of N_l and b_l in the corresponding vacuum state are then determined by the periods of the holomorphic differential,

$$T = \left\langle \mathrm{Tr}_N \left[\frac{dx}{x-\Phi}\right]\right\rangle = \frac{P_N'(x)}{\sqrt{P_N(x)^2 - 4\Lambda^{2N}}}\,dx \tag{4.9}$$

as,

$$N_l = \oint_{A_l} T \qquad -\tau_0 - b_l = \oint_{B_l} T\,. \tag{4.10}$$

Note that the periods of T around the compact cycles A_l and $B_k - B_l$ are integer-valued as expected. This follows immediately from the relation [25],

$$T = -d\log\left(P_N(x) - \sqrt{P_N(x)^2 - 4\Lambda^{2N}}\right). \tag{4.11}$$

In the language of the dual string theory, the periods of T around any given one-cycle on Σ measure the number of units of RR three-form flux through the corresponding three-cycle of the Calabi-Yau geometry (3.9). The first relation in (4.10) confirms that there are N_l units of flux through the basis three-cycles S^3_l. The second relation tells us that the RR flux through the compact linking cycles discussed in section 3.4 is determined by the integers b_l.

The problem of factorising the curve to find an on-shell realisation of the critical behaviour described above was solved in [43, 44]. The desired factorisation can only be obtained when N is divisible by m. In this case we set $\kappa_1 = 4\varepsilon^2\Lambda^{2m}$ in (4.2) with $\kappa_l = 0$ for $l > 1$ as before. Thus the spectral curve becomes,

$$y^2 = \varepsilon^2(x^m - \mathcal{U})^2 - 4\varepsilon^2\Lambda^{2m} = \varepsilon^2(x^m - \mathcal{U} - 2\Lambda^m)(x^m - \mathcal{U} + 2\Lambda^m)\,. \tag{4.12}$$

The on-shell curve becomes singular at $\mathcal{U} = \pm 2\Lambda$ and coincides with the critical curve (4.4). In particular, choosing the positive sign we find the curve,

$$y^2 = \varepsilon^2 x^m (x^m - 4\Lambda^m) \tag{4.13}$$

with the singularity $y^2 \sim x^m$ at the origin.

The values of the integers N_l and b_l corresponding to this vacuum can then be computed using (4.10). In this way we obtain $N_1 = N_2 = \ldots N_m = n$ where $N = mn$ and $b_1 = b_2 = \ldots b_m = 0$. The first condition means that the unbroken classical gauge group in this vacuum $\hat{G}_{cl}$ is $\mathrm{U}(n)^m$ and the second implies that the mode of confinement in the non-abelian part of each factor is the same. By the formula of section 2, the confinement index in this vacuum is given by $t = n$. It is striking that the critical curve (4.4), (4.13) can only be realised on-shell in this specific vacuum. In particular, the restriction on b_l is essential to ensure that the charged states which become massless at the critical point are present in the theory as unconfined particles.

In the context of $\mathcal{N} = 2$ supersymmetry, the singular curve (4.13) corresponds to a special point on the Coulomb branch of $\mathcal{N} = 2$ SUSY Yang-Mills theory with gauge group $\mathrm{SU}(m)$ where a number of mutually non-local electric and magnetic charges become simultaneously massless (for $m > 2$). In this vacuum the theory flows to the a_{m-1} Argyres-Douglas superconformal fixed point [31]. The present context is somewhat different. We are considering a $\mathrm{U}(N)$ gauge theory in a partially confining phase with $\mathcal{N} = 1$ supersymmetry. The degeneration of the spectral curve which controls the coupling constants of the low-energy gauge group $\hat{G} = \mathrm{U}(1)^m$ is consistent with the presence of the same set of massless electric and magnetic charges which appear at the Argyres-Douglas fixed point. In our case these states are identified as electric dibaryons and their magnetic (and dyonic) counterparts as discussed above. It has been proposed [43] that the resulting $\mathcal{N} = 1$ theory flows to the *same* $\mathcal{N} = 2$ superconformal fixed point in the IR. Evidence in favour of this includes the spectrum of chiral operator dimensions, essentially determined by the curve (4.13), which matches that of the Argyres-Douglas CFT. Although we believe this proposal is plausible we will not assume it to be true in the following.

The above points are further illuminated by considering the behaviour of the dual string geometry at the critical point. For $\mathcal{U} = 2\Lambda^m$ (3.9) becomes

$$u^2 + v^2 + w^2 + \varepsilon^2 x^m (x^m - 4\Lambda^m) = 0 \tag{4.14}$$

which exhibits a generalised conifold singularity of type a_{m-1} at the point $u = v = w = x = 0$. The case of $\mathcal{N} = 2$ supersymmetry described above corresponds to considering IIB strings on the singular Calabi-Yau geometry (4.14) *without flux*. The low-energy effective action for string theory in this background includes the $\mathcal{N} = 2$ Argyres-Douglas SCFT living in the remaining four flat dimensions of the ten dimensional spacetime. As above, the massless electric and magnetic states correspond to D3 branes wrapped on three-cycles which vanish at the critical point. The resulting four-dimensional superconformal theory can be decoupled from gravity by taking the limit $g_s \to 0$. We will review the standard analysis of this flux-free case in more detail in section 6.1 below.

As in our field theory discussion, the present context is somewhat different. In particular the geometry includes non-trivial RR flux through certain three-cycles. To be precise, as we have $N_l = n = N/m$ for all l, the basis three-cycles which project to the one-cycles A_l on Σ each have exactly n units of flux. This flux introduces a superpotential for the moduli of the Calabi-Yau which breaks the spacetime supersymmetry to $\mathcal{N} = 1$ in four dimensions. On the other hand, as the integers b_l vanish in our chosen vacuum, there is no flux through the compact three-cycles which project to compact one-cycles of the form $B_k - B_l$ on Σ. In particular, this means that there is no flux through the cycles which vanish at the critical point.[12] As mentioned above this means that the theory contains unconfined charged states corresponding to D3 branes wrapped on these cycles. This in turn is consistent with the appearance at the critical point of the same set of massless charges associated with the $\mathcal{N} = 2$ Argyres-Douglas fixed point as suggested above.

5. The double-scaling limit

In the previous section we gave a general argument that the $1/N$ expansion should break down at critical points where the electric and magnetic dibaryon become massless. We can now illustrate this breakdown by considering the behaviour of the planar diagram expansion (3.8) for the three-point coupling $V^{(3)}$ as we approach the critical point. As above we begin by considering the off-shell behaviour on the submanifold in field space defined by setting $S_l = S\exp(2\pi i l/m)$, for $l = 1, 2, \ldots, m$. In this case (3.8) simplifies to,

$$V^{(3)}_{ijk} = \frac{1}{\varepsilon\mathcal{U}^{\frac{m+1}{m}}} \sum_{r=0}^{\infty} c^{(r)}_{ijk} \left(\frac{S}{\varepsilon\mathcal{U}^{\frac{m+1}{m}}}\right)^r \tag{5.1}$$

with coefficients $c^{(r)}_{ijk}$ coming from summing the planar diagrams with $r+2$ loops. The exact answer for the sum (5.1) is then determined by equations (3.5), (2.13) and (4.2). Near the critical point we obtain,[13]

$$\begin{aligned} V^{(3)}_{ijk} &\sim \frac{1}{\varepsilon\mathcal{U}^{\frac{m+1}{m}}} \left(1 - \frac{1}{d_m}\left(\frac{S}{\varepsilon\mathcal{U}^{\frac{m+1}{m}}}\right)\right)^{-1} \\ &\sim \frac{1}{S - S_{cr}} \end{aligned} \tag{5.2}$$

As above we can realise this behaviour on-shell by setting $\kappa_1 = 4\varepsilon^2\Lambda^{2m}$. The critical behaviour is then attained at $\mathcal{U} = 2\Lambda^m$. Near this point in parameter space, (5.2) is equivalent to,

$$V^{(3)}_{ijk} \sim \frac{1}{\varepsilon\Lambda^{\frac{m}{2}}(\mathcal{U} - 2\Lambda^m)^{\frac{m+2}{2m}}} \,. \tag{5.3}$$

[12] This can also be understood by noting that a vanishing cycle with a non-zero quantised flux automatically leads to to a divergence in the RR field strength. This would correspond to a divergence in the superpotential and its derivatives which is not consistent with a SUSY vacuum for finite values of the couplings.

[13] The formulae (5.2) and (5.3) below are meant to indicate the leading behaviour of the most divergent elements of the tensor $V^{(3)}_{ijk}$. We will give much more precise results in the next section.

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These results indicate that the series of planar diagrams (3.8) has a finite radius of convergence and, in particular, it diverges at the critical point $S_l = S_{cr}\exp(2\pi i l/m)$. This type of behaviour is very familiar in the context of matrix models and is hardly surprising given that the planar diagrams contributing to the F-term couplings reduce to those of an auxiliary matrix model. However, the important point is that the divergence reflects the presence of new light states at the critical point with couplings to the glueballs which remain unsuppressed in the large-N limit. This means that similar divergences should also occur in generic correlation functions of single trace operators.

In the context of zero dimensional matrix models, the type of critical behaviour described above can be used to define a double-scaling limit (for a review see [14]). Near the critical point, the sum of planar diagrams is dominated by large diagrams with many vertices. Related divergences also occur in the sum over diagrams of each fixed genus corresponding to contributions at each order in the $1/N$ expansion. By taking $N \to \infty$ and simultaneously tuning the parameters to approach the critical point, one can cure all of these divergences simultaneously and obtain a coherent sum over large diagrams of each genus. The sum over different genera is then controlled by a renormalised expansion parameter κ which is held fixed as $N \to \infty$. A characteristic feature of the resulting double-scaled theories is the existence of universality classes associated with each critical point. After taking the limit, detailed information about the definition of the theory is lost and seemingly different models become equivalent.

Double-scaled matrix models also have an interesting interpretation in terms of a dual string theory. In this context individual Feynman diagrams correspond to discretisations of the string world-sheet. The double-scaling limit is then equivalent to a continuum limit on the worldsheet. In many cases, this yields a continuum non-critical string theory. Typically the dilaton grows in the Liouville direction but this growth is cut-off by the presence of a potential term in the world sheet action (the "Liouville wall"). The resulting effective string coupling is identified with the double scaling parameter κ. In the following, we will define a corresponding double-scaling limit for our four-dimensional theory where very similar features emerge.

In the standard large-N 't Hooft limit discussed in section 3, the interaction vertices of L single trace fields, go to zero like N^{2-L}. Motivated by the above discussion, we can instead define a large-N double-scaling limit where these interactions are held fixed. As before we take $N \to \infty$ with $\varepsilon \sim N$ and Λ fixed. The new ingredient is that we simultaneously take the limit $\mathcal{U} \to 2\Lambda^m$ with,

$$\Delta = \varepsilon(\mathcal{U} - 2\Lambda^m)^{\frac{m+2}{2m}} \tag{5.4}$$

held fixed. From (5.3), we see that the three-point vertex $V^{(3)}_{ijk} \sim (1/\Lambda^{m/2}\Delta)$ remains fixed in this limit. We will analyse the effect of this limit on the rest of the F-term effective action below. After allowing for the generalisation (5.5) discussed below, this is the unique limit which leads to a finite non-zero interaction between colour-singlet states and also keeps the mass scale of the glueball spectrum fixed. As discussed above, the breakdown of the $1/N$ expansion can also be understood as a consequence of the presence of light $\hat{G}$-charged states near the critical point. In particular the loop contribution (4.1) suggests that the effective coupling for glueball interactions is proportional to $1/M_B$ where M_B is the mass of the lightest charged dibaryon. Combining these observations, we conclude that the double-scaling limit proposed above also has the effect of keeping M_B constant. In the following, several checks of the consistency of this conclusion will emerge.

In the context of zero dimensional matrix models, one of the key features of the double-scaling limit is that it yields finite non-zero contributions at each genus. This should also be true in four-dimensions if the limit defined above holds the dibaryon mass fixed as we claim. Ideally one could check this at each order by considering higher genus corrections to the F-term effective action. However the effective superpotential is saturated by planar diagrams and it receives no $1/N$ corrections. On the other hand, such corrections do appear when the four-dimensional field theory is coupled to gravity and a background field-strength for the graviphoton is turned on. These corrections are in turn related to the higher genus corrections to the free energy of the auxiliary matrix model. Although we will not discuss this in detail here, the above discussion suggests that the limit defined above should correspond to a double-scaling limit of the auxiliary matrix model which yields a finite contribution at each genus. This is hard to verify directly, as very little is known about higher genus corrections for the multi-cut configurations of the complex matrix model which are relevant to the critical point of interest. On the other hand, the conjectured duality between matrix models and topological string theory developed in [19 − 21] relates these corrections to the higher-genus contributions to the B-model on the Calabi-Yau geometry (3.9). Possible tests of our proposal along these lines will be discussed elsewhere. Previous work [19, 23, 42] on double-scaling limits of the Dijkgraaf-Vafa matrix model and its four-dimensional interpretation has focussed on cases which can be analysed using known results for one- and two-cut solutions of the hermitian matrix model. The analysis of the two-cut solution given in the [23, appendix] should be relevant to the special case $m = 2$ of our analysis.

It will also be useful to generalise the double-scaling limit slightly by restoring the full set of superpotential couplings g_l appearing in (2.1) for $l = 1, 2, \ldots, m$. As before the critical point lies at $g_1 = \mathcal{U} = 2\Lambda^m$ with $g_l = 0$ for $l > 1$. We approach the critical point by taking the limit $N \to \infty$ and $\varepsilon \to \infty$ with $\delta = (g_1 - 2\Lambda^m) \to 0$ and $g_l \to 0$. Apart from Λ, the quantities held fixed are,

$$\tilde{\varepsilon} = \frac{\varepsilon}{N} \qquad \Delta = \varepsilon\delta^{\frac{m+2}{2m}} \qquad \textit{and} \qquad \tilde{g}_l = \frac{g_l}{\delta^{\frac{m-l+1}{m}}} \quad \text{for} \quad l = 2, \ldots, m\,. \tag{5.5}$$

In this more general limit, the three-point function has the behaviour,

$$V^{(3)}_{ijk} \sim \frac{1}{\Lambda^{\frac{m}{2}}\Delta}\, v^{(3)}_{ijk}(\tilde{g}_l) \tag{5.6}$$

The tensor $v^{(3)}_{ijk}$ is a function of the additional double-scaling parameters $\tilde{g}_l$ which will be determined explicitly below.

In order to understand the behaviour of the theory in the double-scaling limit described above, it is convenient to exploit the electric-magnetic duality invariance of the F-term action (2.10). In particular, the $\mathcal{N} = 1$ superspace action $W^{(2)}_{\text{eff}}$ for the low-energy abelian vector multiplets $w_{\alpha l}$ is invariant under the Sp$(2m;\mathbb{Z})$ low-energy S-duality transformations which are familiar from the study of $\mathcal{N}=2$ supersymmetric gauge theory [46]. As we review below, this invariance extends to the full F-term action with an appropriate transformation law for the integers N_l and b_l appearing in $W_{\text{eff}}^{(0)}$. For technical reasons we will restrict our attention to the case of odd m setting $m=2p+1$. We will comment on the case of even m at the end of this section.

In terms of the spectral curve Σ, these electric-magnetic duality transformations correspond to our freedom in choosing a basis of homology cycles with the canonical intersection properties (2.12). Starting from the basis of cycles, A_l and B_l, with $l=1,2,\ldots,m$, defined in the previous section, we can find a new canonical basis with cycles $\tilde{A}_l$ and $\tilde{B}_l$ where,

$$\begin{pmatrix} \tilde{B} \\ \tilde{A} \end{pmatrix} = M \cdot \begin{pmatrix} B \\ A \end{pmatrix} \tag{5.7}$$

for $M \in \text{Sp}(2m;\mathbb{Z})$.

We will choose to work in a very specific basis which we now describe. We define basis cycles,

$$\begin{aligned} \tilde{A}_l &= A_-^{(\kappa)} & \tilde{B}_l &= B_-^{(\kappa)} & & l=\kappa=1,2,\ldots,p \\ &= A_+^{(\kappa)} & &= B_+^{(\kappa)} & & l=p+\kappa=p+1,p+2,\ldots,2p \\ &= A_\infty & &= B_\infty & & l=2p+1 \end{aligned}$$

Apart from having canonical intersections the new basis cycles also have the following properties,

1. The cycles $A_-^{(\kappa)}$ and $B_-^{(\kappa)}$ for $\kappa=1,2,\ldots,p$ are compact cycles which vanish at the critical point.

2. The cycles $A_+^{(\kappa)}$ and $B_+^{(\kappa)}$ for $\kappa=1,2,\ldots,p$ are compact cycles which have zero intersection with all the vanishing cycles.

3. The compact cycle $A_\infty = A_1 + A_2 + \cdots + A_m$ corresponds to a large circle in the complex x-plane enclosing all of the branch cuts. B_∞ is a non-compact cycle which necessarily has zero intersection with all the vanishing cycles.

The new basis is illustrated for the case $m=3$ in figure 4. The corresponding Sp$(6;\mathbb{Z})$ transformation is given in the appendix. It should be clear by inspection that such a choice is possible for any odd value of m. In terms of the dual string theory, the Sp$(2m;\mathbb{Z})$ transformation corresponds to a change of basis for the third homology group of the Calabi-Yau manifold (3.9). In the new basis, the compact three-cycles corresponding to $A_-^{(\kappa)}$ and $B_-^{(\kappa)}$, for $\kappa=1,2,\ldots,p$ all vanish at the generalised conifold singularity, while the remaining three-cycles are non-vanishing.

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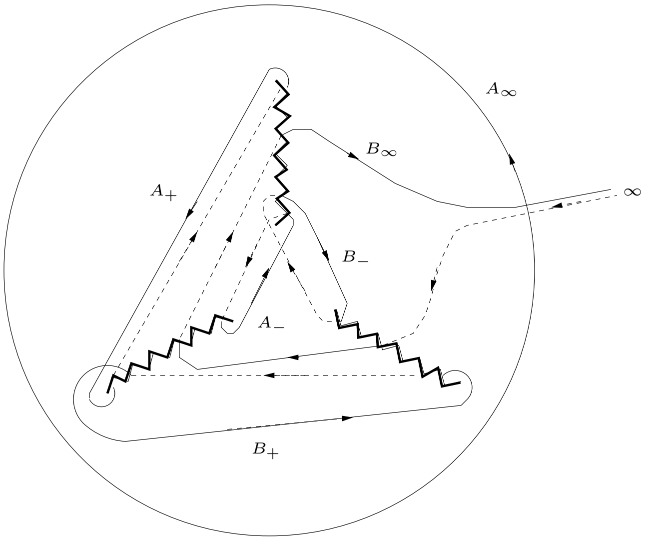

Figure 4: The new basis for $m = 3$.

The change of basis described above induces a non-trivial transformation of the glueball superfields S_l and the conjugate variables S_l^D. These quantities are given in (2.13) above as A- and B-periods of the holomorphic differential ydx. Thus we have,

$$\begin{pmatrix} \tilde{S}^D \\ \tilde{S} \end{pmatrix} = M \cdot \begin{pmatrix} S^D \\ S \end{pmatrix} \tag{5.8}$$

where,

$$\tilde{S}_l = \oint_{\tilde{A}_l} ydx \qquad \tilde{S}_l^D = \frac{1}{2\pi i}\frac{\partial \tilde{\mathcal{F}}}{\partial \tilde{S}_l} = \oint_{\tilde{B}_l} ydx \tag{5.9}$$

This transformation allows us to rewrite the F-term effective action in terms of the new fields,

$$\tilde{S}_l = -\frac{1}{2\pi i}\oint_{\tilde{A}_l} dx\, \frac{1}{32\pi^2}\mathrm{Tr}_N\left[\frac{W_\alpha W^\alpha}{x-\Phi}\right] \tag{5.10}$$

The resulting action for the dual vector multiplets $\tilde{w}_{\alpha l}$ is determined by a dual prepotential $\tilde{\mathcal{F}}(\tilde{S}_1, \ldots, \tilde{S}_m)$ which is defined by the second equality in (5.9). The corresponding contribution to the superpotential is,

$$W_{\rm eff}^{(2)} = \frac{1}{2}\sum_{k,l}\frac{\partial^2 \tilde{\mathcal{F}}}{\partial \tilde{S}_k \partial \tilde{S}_l}\,\tilde{w}_{\alpha k}\tilde{w}_l^\alpha \tag{5.11}$$

After expanding the action in component fields as in (3.4), we find L-point interaction vertices,

$$\tilde{V}^{(L)}_{i_1 i_2 \ldots i_L} = \left\langle \frac{\partial^L \tilde{\mathcal{F}}}{\partial \tilde{S}_{i_1}\partial \tilde{S}_{i_2}\ldots \partial \tilde{S}_{i_L}}\right\rangle \tag{5.12}$$

for $L = 2, 3, 4$.

To rewrite the glueball superpotential $W_{\rm eff}^{(0)}$ in terms of the new variables, we need to consider the action of $\mathrm{Sp}(2m;\mathbb{Z})$ on the integers N_l and b_l corresponding to our choice of vacuum. The quantities N_l and $N_l^D = -\tau_0 - b_l$ are given by periods of the holomorphic differential T and therefore transform as,

$$\begin{pmatrix} \tilde{N}^D \\ \tilde{N} \end{pmatrix} = M \cdot \begin{pmatrix} N^D \\ N \end{pmatrix} \tag{5.13}$$

where,

$$\begin{aligned} \tilde{N}_l &= \oint_{\tilde{A}_l} T \qquad \tilde{N}_l^D = -\tilde{b}_l = \oint_{\tilde{B}_l} T \qquad l = 1, \ldots, 2p \\ \tilde{N}_{2p+1} &= \oint_{\tilde{A}_\infty} T = N \qquad \tilde{N}_{2p+1}^D = -\tilde{\tau}_0 - \tilde{b}_{2p+1} = \oint_{\tilde{B}_\infty} T \, . \end{aligned} \tag{5.14}$$

The bare coupling transforms as $\tilde{\tau}_0 = \tau_0 + q$ for some integer q, which corresponds to an irrelevant shift of the bare vacuum angle by a multiple of 2π. The glueball superpotential has the form of a simplectic product between period vectors and is therefore invariant under the $\mathrm{Sp}(2m;\mathbb{Z})$ duality transformation. In the new basis it can be written as,

$$W_{\mathrm{eff}}^{(0)} = \sum_{l=1}^{2p} \left(\tilde{N}_l \frac{\partial \tilde{\mathcal{F}}}{\partial \tilde{S}_l} + 2\pi i \tilde{b}_l \tilde{S}_l \right) + \tilde{N}_{2p+1} \frac{\partial \tilde{\mathcal{F}}}{\partial \tilde{S}_{2p+1}} + 2\pi i (\tilde{\tau}_0 + \tilde{b}_{2p+1}) \tilde{S}_{2p+1} \, . \tag{5.15}$$

The complete effective superpotential in the new basis is given by the sum of (5.11) and (5.15).

We now present our results for the behaviour of the F-term couplings in the double-scaling limit. Details of the calculation are given in the appendix. We begin by considering the two point coupling $\tilde{V}_{ij}^{(2)}$ which determines the matrix of low-energy abelian gauge couplings. As above, this object is a generalised period matrix for the (non-compact) Riemann surface Σ. In the double-scaling limit, we find that the period matrix assumes the following block-diagonal structure,

$$\tilde{V}^{(2)} = \begin{pmatrix} \Pi^- & 0 \\ 0 & \Pi^+ \end{pmatrix} . \tag{5.16}$$

Here Π^- is a $p \times p$ matrix-valued function of the moduli $\tilde{g}_l$ where, as above, $p = (m-1)/2$. This matrix can be identified with the period matrix of the *reduced spectral curve* Σ_-. The latter is a compact Riemann surface of genus p described by the complex equation,

$$\mathcal{Y}^2 = \mathcal{X}^m - \sum_{l=2}^{p} \tilde{g}_l \mathcal{X}^{l-1} - 1 \, . \tag{5.17}$$

The remaining $q \times q$ block Π^+ (with $q = m - p$) can be thought of as the generalised period matrix of the non-compact Riemann surface Σ_+ described by the curve,

$$y^2 = \varepsilon^2 (x^m - 4\Lambda^m) \, . \tag{5.18}$$

The block diagonalisation of the generalised period matrix described above corresponds to a particular degeneration of the spectral curve Σ occurring in the double-scaling limit. A simple intuitive picture of the degeneration is that Σ decomposes into a union of the

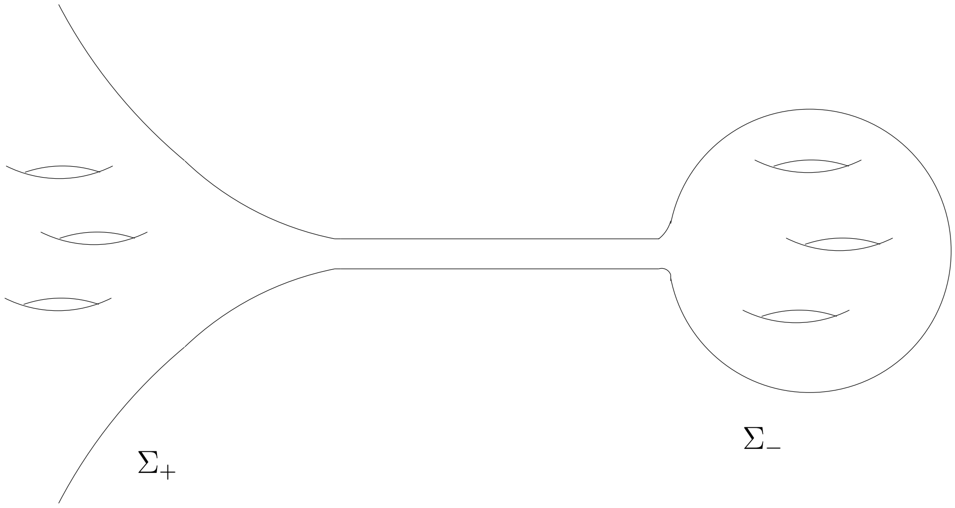

Figure 5: The degeneration of the spectral curve Σ in the double-scaling limit.

two Reimann surfaces Σ_- and Σ_+ defined above joined together by a long, thin tube as shown in figure 5. A similar degeneration in the case $m = 3$ was discussed in [31]. Such an interpretation must be treated with care as it refers to a particular choice of metric on the surface Σ. At this point we know the complex structure of Σ which only determines the metric up to conformal transformations. Despite this, we will argue in the next section that the picture of the Riemann surface degenerating into two sectors linked by a long tube or throat is directly relevant in the context of the string theory dual.

The couplings in the F-term effective action reflect the splitting of the period matrix described above. In particular, the three-point coupling $\tilde{V}^{(3)}_{ijk}$ is only non-zero in the double-scaling limit if all three indices are constrained to take values less than or equal to $p = (m-1)/2$. Thus we find that,

$$\tilde{V}^{(3)}_{ijk} = \frac{1}{\Lambda^{\frac{m}{2}}\Delta} v_{\alpha\beta\delta}(\tilde{g}_l) \tag{5.19}$$

for $i = \alpha = 1, 2, \ldots, p$, $j = \beta = 1, 2, \ldots, p$, $j = \delta = 1, 2, \ldots, p$ and vanishes identically in the double-scaling limit otherwise. Similar results apply to the four-point coupling $\tilde{V}^{(4)}$.

It is also interesting to consider the behaviour of the glueball superpotential $W^{(0)}_{\rm eff}$, as given in (5.15), in the double-scaling limit. This depends explicitly on the integers $\tilde{N}_l$ and $\tilde{b}_l$ which determine our choice of vacuum. These may be evaluated by calculating the periods of the differential form T on the new basis cycles $\tilde{A}_l$ and $\tilde{B}_l$. Alternatively, we can apply the $\mathrm{Sp}(2m;\mathbb{Z})$ transformation (5.13) to the corresponding integers $N_1 = N_2 = \ldots = N_m = n$ and $b_1 = b_2 = \ldots = b_m = 0$ in the original basis. By either method we find that $\tilde{N}_l = \tilde{b}_l = 0$ for $l = 1, 2, \ldots, p$. In terms of the string dual, this is equivalent to the statement that there is no RR flux through those three-cycles which vanish at the generalised conifold singularity. Thus (5.15) simplifies to become,

$$W^{(0)}_{\rm eff} = 2\pi i \sum_{l=p+1}^{2p} \left(\tilde{N}_l \tilde{S}^D_l + \tilde{b}_l \tilde{S}_l\right) + 2\pi i \tilde{N}_{2p+1} \tilde{S}^D_{2p+1} + 2\pi i (\tilde{\tau}_0 + \tilde{b}_{2p+1}) \tilde{S}_{2p+1}\,. \tag{5.20}$$

In equation (5.20), the superpotential is expressed purely in terms of periods of ydx around one-cycles which have zero intersection with all the vanishing one-cycles. In terms of the diagonal decomposition of the period matrix illustated in figure 5, these correspond to periods of the surface Σ_+. As one might expect, these have only weak dependence on the moduli of the surface Σ_-.

$$\tilde{H}^{(2)}_{ij} = \frac{\partial W^{(0)}_{\text{eff}}}{\partial \tilde{S}_i \partial \tilde{S}_j} \tag{5.21}$$

which is given by,

$$\tilde{H}^{(2)}_{ij} = \frac{n}{\varepsilon \mathcal{U}^{\frac{1}{2}}} h^{(2)}_{AB} \tag{5.22}$$

for $i = p + A = p + 1, \ldots 2p + 1$, $j = p + B = p + 1, \ldots 2p + 1$ and vanishes identically in the double scaling limit otherwise. Here $h^{(2)}_{AB}$ is a constant $q \times q$ matrix. Thus $\tilde{H}^{(2)}$ has precisely the opposite behaviour of the three-point coupling, namely it is only non-zero in the $q \times q$ block corresponding to the period matrix Π^+.

With some additional assumptions, the vanishing of the complimentary $p \times p$ block corresponding to Π_- indicates the emergence of p massless glueballs in the double-scaling limit. In particular, if we make the standard assumption that the glueball fields $\tilde{S}_l$ can be treated as independent fields with non-singular kinetic terms then we can infer that the theory contains p neutral massless chiral multiplets of $\mathcal{N} = 1$ supersymmetry in the double-scaling limit. The occurrence of a massless glueball in the case $m = 2$ was noted in [42]. Beyond this, one can also check that all higher derivatives of the superpotential where even one of the derivatives is with respect to a field $\tilde{S}_l$ with $l \leq p$ vanish in the double-scaling limit. This means that all the F-term interaction vertices for these fields coming from the glueball superpotential vanish in the double-scaling limit.

There is a simple way to summarise the results described in this section which is inspired by the intuitive picture of the spectral curve Σ degenerating into curves Σ_+ and Σ_- joined by a long thin tube as shown in figure 5. The main points are,

1. The scaling of the couplings in the F-term effective action are consistent with the existence of two sectors $\mathcal{H}_\pm$ in the Hilbert space of the theory which decouple in the large-N double-scaling limit. One sector, $\mathcal{H}_-$, contains $p = (m-1)/2$ massless U(1) vector multiplets of $\mathcal{N} = 1$ SUSY, $\tilde{w}_{\alpha l}$, together with p neutral chiral multiplets $\tilde{S}_l$ for $l = 1, 2, \ldots, p$. Our result (5.22) indicates that the chiral multiplets also become massless in the double-scaling limit. Standard large-N reasoning suggests that this sector should also contain many additional massive degrees of freedom. These include for example, infinite towers of massive single-particle states created by the single trace operators $\tilde{S}_l$. The other sector $\mathcal{H}_+$ contains the remaining $q = m - p$ massless U(1) vector multiplets as well as other massive degrees of freedom.

2. Interactions between colour-singlet states in the sector $\mathcal{H}_-$ are controlled by the coupling,

$$\frac{1}{N_{\text{eff}}} \sim \frac{1}{\Delta} \tag{5.23}$$

which is held fixed in the double-scaling limit.[14] These interactions also depend on the remaining double-scaling parameters $\tilde{g}_l$ for $l = 2, \dots p$. Interactions between colour-singlet states in the sector $\mathcal{H}_+$ are suppressed by powers of $1/N$ as in a conventional large-N 't Hooft limit. Interactions between states in the sector $\mathcal{H}_+$ and those in the sector $\mathcal{H}_-$ also go to zero as $N \to \infty$.

3. In the double-scaling limit, the F-term effective action for the fields in sector $\mathcal{H}_-$ is determined by the reduced spectral curve Σ_- defined in (5.17) above. In particular we can find a canonical basis of one-cycles $\mathcal{A}_l$ and $\mathcal{B}_l$ for $l = 1, 2, \dots p$ such that,

$$\tilde{S}_l = 2i\Lambda^{\frac{m}{2}}\Delta \oint_{\mathcal{A}_l} \mathcal{Y} d\mathcal{X} \qquad \tilde{S}_l^D = \frac{1}{2\pi i}\frac{\partial \mathcal{F}_-}{\partial \tilde{S}_l} = 2i\Lambda^{\frac{m}{2}}\Delta \oint_{\mathcal{B}_l} \mathcal{Y} d\mathcal{X} \tag{5.24}$$

which defines a reduced prepotential $\mathcal{F}_-[\tilde{S}_1, \dots, \tilde{S}_p]$. The complete F-term action for the sector $\mathcal{H}_-$ in the double-scaling limit can then be written as,

$$\mathcal{L}_F^- = \mathrm{Im}\left[\int d^2\theta \sum_{k,l\leq p} \frac{1}{2}\frac{\partial^2 \mathcal{F}_-}{\partial \tilde{S}_k \partial \tilde{S}_l} \tilde{w}_{\alpha k}\tilde{w}_l^{\alpha}\right]. \tag{5.25}$$

Note, in particular that the effective superpotential for the massless fields $\tilde{S}_l$, $l \leq p$, is exactly zero reflecting the exact vanishing of the interaction vertices for these fields described above.

In this section we have restricted our attention to the case of odd m. The additional complication of the case of even m stems from the presence of an additional holomorphic differential which is log-normalisable at the singularity. In this case we find an interacting sector of $p = m/2-1$ massless vectors and p massless neutral chirals.[15] The non-interacting sector contains $q = m - p$ massless free vectors together with $q - 1$ massive chirals. The theory also contains one extra massless chiral multiplet which is also free. Note that the interacting sector has no massless fields for the special case $m = 2$ which corresponds to the ordinary conifold. These results will be reported in more detail elsewhere.

6. The dual string theory

In this section we will interpret the results of our field theory calculation in terms of the dual string theory. As a preliminary, we will begin by reviewing some standard facts about Type IIB string theory on the generalised conifold geometry without flux.

6.1 The generalised conifold without flux

The generalised conifold geometry has the form,

$$U^2 + V^2 + W^2 + Z^m = 0\,. \tag{6.1}$$

[14] To make the coupling dimensionless we could instead write $1/N_{\rm eff} \sim \Lambda^{3-\frac{m}{2}}/\Delta$ where $\sim$ denotes equality up to an unknown dimensionless constant of order N^0

[15] As above we define $p = [(m-1)/2]$ for all values of m.

String amplitudes on this background are singular at each order in string perturbation theory. The singularity shows up in a linear σ-model formulation of the world-sheet theory [47], where a new non-compact branch opens up apparently leading to an infinite number of massless states. When non-perturbative effects are included the picture is somewhat different. As we now review, the singularity can be understood in terms of a finite number of massless states corresponding to D3 branes wrapped on vanishing three-cycles [48].

It is convenient to consider first the non-singular geometry obtained when we slightly deform the complex structure of the Calabi-Yau (6.1) to,[16]

$$F(U,V,W,Z) = U^2 + V^2 + W^2 + Z^m - \sum_{l=1}^{p} \mu_l Z^{l-1} = 0\,, \tag{6.2}$$

where $p = [(m-1)/2]$ as above. The restriction to monomials Z^k with $k \leq p$ corresponds to considering only normalisable deformations of the geometry [57]. Each normalisable deformation leads to an $\mathcal{N}=2$ vector multiplet in the low-energy four-dimensional effective theory. Thus the low-energy theory has gauge group $\mathrm{U}(1)^p$. The effect of this deformation is to blow up various three-cycles to finite size. The theory now contains contains massive BPS states corresponding to D3 branes wrapped on these cycles. These states carry electric and magnetic charges under the low-energy $\mathrm{U}(1)^p$ gauge symmetry. Their BPS mass formula has the form,

$$M_{\mathrm{D3}} \sim \frac{1}{g_s\sqrt{\alpha'}} \left| \int_{S^3} \Omega \right|, \tag{6.3}$$

where

$$\Omega = \frac{dU\, dV\, dW\, dZ}{dF} \tag{6.4}$$

is the holomorphic three-form of the Calabi-Yau geometry (6.2). By setting $\mu_l = 0$, the wrapped D3 branes become massless and we return to the generalised conifold singularity (6.1). It is believed that the effect of the light D-branes on the target space geometry is to replace the region close to the singularity by an infinite throat with a linear dilaton [49, 50]. As the dilaton grows, the effective string coupling becomes large and string perturbation theory breaks down.

Another effect related to the appearance of an infinite throat is that the strongly interacting massless degrees of freedom which appear at the singularity can be decoupled from gravity. In particular, taking the limit $g_s \to 0$ makes the ten dimensional bulk theory away from the singularity free. States localised at the singularity can retain finite interactions in this limit precisely because of the linear growth of the dilaton in the throat. The resulting interacting decoupled theory includes the p massless vector multiplets and the mutually non-local (for $m > 2$) charged states corresponding to the wrapped D3 branes. In additional to these massless modes, the theory also contains a full set of massive stringy excitations corresponding to bound states of the fundamental IIB string localised at the singularity. The interactions of these localised strings are described by a non-critical string

[16] We are using conventions where the dimensionless space-time coordinates U, V, W and Z measure distance in units of the string length $\sqrt{\alpha'}$. Consequently the deformation parameters μ_l are also dimensionless.

theory in four dimensions without gravity, also known as a Little String Theory (LST). Like the IIB background we started with, the LST has $\mathcal{N} = 2$ supersymmetry in four dimensions. Far below the string scale, only the massless degrees of freedom are relevant and the interactions of the vector multiplets and mutually non-local charges are described by the Argyres-Douglas $\mathcal{N} = 2$ superconformal field theory of type a_{m-1}.

As in other examples, the dynamics of LST can be usefully studied using holographic duality. In general, non-local theories without gravity of this type are dual to "ordinary" critical string theory on a linear dilaton background [15]. In the present case, the LST of the generalised conifold is holographically dual to the IIB string propagating in the infinite throat region described above. This description is still of limited use because the effective string coupling becomes large in one region of the spacetime. However, as suggested in [16, 8], one may eliminate the strong-coupling region by an appropriate deformation of the string world-sheet theory. In fact, the required deformation simply corresponds to reintroducing the deformation parameters μ_l. As above this has the effect of giving non-zero masses to the BPS states corresponding to wrapped D3 branes. One then takes a double-scaling limit where $g_s \to 0$ and $\mu_l \to 0$ in such a way that the D3 brane masses are held fixed.

To describe the double scaling limit of Giveon and Kutasov in more detail, it is convenient to rescale the spacetime coordinates in (6.2) according to,

$$U = \tilde{U}\mu_1^{\frac{1}{2}} \qquad V = \tilde{V}\mu_1^{\frac{1}{2}} \qquad W = \tilde{W}\mu_1^{\frac{1}{2}} \qquad Z = \tilde{Z}\mu_1^{\frac{1}{m}} \,. \tag{6.5}$$

In this way we find that the masses (6.3) of wrapped D3 branes are given by,

$$M_{\mathrm{D3}} \sim \frac{\mu_1^{\frac{m+2}{2m}}}{g_s\sqrt{\alpha'}} \left| \int_{S^3} \tilde{\Omega} \right| , \tag{6.6}$$

where $\tilde{\Omega} = d\tilde{U}d\tilde{V}d\tilde{W}d\tilde{Z}/d\tilde{F}$ is the holomorphic three-form of the rescaled geometry,

$$\tilde{F}(\tilde{U},\tilde{V},\tilde{W},\tilde{Z}) = \tilde{U}^2 + \tilde{V}^2 + \tilde{W}^2 + \tilde{Z}^m - \sum_{l=2}^{p} \tilde{\mu}_l \tilde{Z}^{l-1} - 1 = 0 \tag{6.7}$$

with $\tilde{\mu}_l = \mu_l / \mu_1^{\frac{m-l+1}{m}}$ for $l = 2, 3, \ldots, p$.

To keep the masses of the wrapped D3 branes constant as $g_s \to 0$, we take the limit $\mu_l \to 0$, with the parameter,

$$\kappa^{-1} = \frac{\mu_1^{\frac{m+2}{2m}}}{g_s} \tag{6.8}$$

held fixed. We also keep, α' and the rescaled parameters $\tilde{\mu}_l$ fixed. The resulting Double-Scaled Little String Theory has an effective string coupling constant $\kappa \sim 1/(M_{D3}\sqrt{\alpha'})$ which remains constant in the limit.

One case where the dual string theory is particularly well understood is the $\mathbb{Z}_m$ symmetric case [8] where $\tilde{\mu}_l = 0$ for $l = 2, 3, \ldots, p$. In this case the string world-sheet theory has the form,

$$R^{3,1} \times \left(\frac{\mathrm{SL}(2)_k}{\mathrm{U}(1)} \times LG(W) \right) / \mathbb{Z}_m \,. \tag{6.9}$$

Here $LG(W)$ denotes the Landau-Ginzburg model with four chiral superfields U, V, W and Z and superpotential,

$$\mathcal{W} = U^2 + V^2 + W^2 + Z^m \,. \tag{6.10}$$

The non-compact coset $\mathrm{SL}(2)_k/\,\mathrm{U}(1)$ corresponds to a semi-infinite cigar geometry with a linear dilaton,

$$\phi = -\frac{Q}{2}\sigma \qquad Q^2 = \frac{2}{k\alpha'} = \frac{m+2}{m\alpha'} \,.$$

The level k of the coset appearing in (6.9) is given by

$$k = \frac{2m}{m+2} \,. \tag{6.11}$$

The effective string coupling has an upper bound given by its value at the tip of the cigar of order

$$g_{\mathrm{cigar}} \sim \kappa \,. \tag{6.12}$$

In the more general case, with $\tilde{\mu}_l \neq 0$ the string world-sheet theory is not known. However, it will be useful to describe the low-energy effective action for the massless degrees of freedom. As above these are p $\mathcal{N}=2$ vector multiplets corresponding to the normalisable complex structure moduli of the Calabi-Yau. In the double-scaling limit where gravity decouples, the low-energy effective action is completely determined by rigid special geometry. The rescaled geometry (6.7) corresponds to an a_1 singularity fibered over a complex curve Γ,

$$\tilde{Y}^2 = \tilde{Z}^m - \sum_{l=2}^{p} \tilde{\mu}_l \tilde{Z}^{l-1} - 1 \,. \tag{6.13}$$

The curve defines a double-cover of the complex Z plane corresponding to a Riemann surface of genus p. Choosing a basis of one-cycles α_l and β_l, for $l = 1, 2, \ldots, p$, with the canonical intersection properties, the vector multiplets a_l and their duals can be expressed as,

$$a_l = \frac{1}{\kappa\sqrt{\alpha'}} \oint_{\alpha_l} \tilde{Y}\, d\tilde{Z} \qquad a_l^D = \frac{1}{2\pi i}\frac{\partial \mathcal{G}}{\partial a_l} = \frac{1}{\kappa\sqrt{\alpha'}} \oint_{\beta_l} \tilde{Y}\, d\tilde{Z} \,, \tag{6.14}$$

where the second relation defines the prepotential $\mathcal{G}$. The BPS mass formula for a wrapped D3 brane with electric/magnetic charges $(\vec{n}_E, \vec{n}_M)$ is,

$$M_{D3} = |n_E^l a_l + n_M^l a_l^D| \,. \tag{6.15}$$

The low energy effective action for the vector multiplets is written as an F-term in $\mathcal{N}=2$ superspace,

$$\mathcal{L}_{\mathrm{eff}} = \mathrm{Im}\left[\int d^4\theta \; \mathcal{G}(a_l)\right] . \tag{6.16}$$

Finally, it will also be useful to rewrite the above results in the language of $\mathcal{N}=1$ supersymmetry. In particular, we can decompose the $\mathcal{N}=2$ vector multiplet a_l, into an $\mathcal{N}=1$ vector multiplet $\mathcal{W}_{\alpha l}$ and a neutral massless chiral multiplet φ_l for $l = 1, 2, \ldots, p$. The low energy effective action (6.16), contains both D-terms and F-terms when rewritten in $\mathcal{N}=1$ superspace. After integrating out half the Grassmann coordinates we find,

$$\mathcal{L}_{\text{eff}} = \text{Im}\left[\int d^2\theta\, \frac{1}{2}\sum_{k,l\leq p} \frac{\partial^2 \mathcal{G}(\varphi_j)}{\partial\varphi_k \partial\varphi_l} \mathcal{W}_{\alpha k}\mathcal{W}_l^\alpha\right] + \text{D} - \text{terms}\,. \tag{6.17}$$

Note that a non-zero superpotential for the chiral multiplets φ_l is forbidden by $\mathcal{N}=2$ supersymmetry.

6.2 A duality proposal

We now return to consider the IIB geometry dual to the four-dimensional gauge theory with superpotential (2.1). In a generic vacuum this has the form,

$$u^2+v^2+w^2+\varepsilon^2\left(x^m - \sum_{l=1}^{m} g_l x^{l-1}\right)^2 - 4\varepsilon^2\Lambda^{2m} =$$
$$u^2+v^2+w^2+\varepsilon^2\prod_{l=1}^{m}(x-x_+^{(l)})(x-x_-^{(l)}) = 0\,. \tag{6.18}$$

At the critical point the m branch-points $x_-^{(l)}$ collide and we find the singular geometry,

$$u^2+v^2+w^2+\varepsilon^2 x^m(x^m - 4\Lambda^m) = 0 \tag{6.19}$$

which has a generalised conifold singularity at the origin.

There are two main differences with the standard discussion of the generalised conifold given in the previous section. First, the singularity appears as part of a larger geometry which also includes non-trivial cycles of finite size of order Λ. Secondly there is RR flux through these cycles. Specifically there is n units of flux through the cycles whose projections on the x-plane surround the cuts running from $x_-^{(l)}$ to $x_+^{(l)}$ for $l=1,2,\ldots m$. The effect of this flux is to introduce a non-trivial superpotential which fixes the moduli of the Calabi-Yau. As discussed by Aganagic, Mariño and Vafa [33], one way to make contact with the flux-free case of the previous section is to take a particular $\Lambda\to\infty$ limit where the non-vanishing cycles become very large and the RR field-strength goes to zero everywhere.

In the following we will suggest a quite different limit which also makes contact with the generalised conifold without flux. Specifically, we will consider the large-N double-scaling limit of the dual field theory defined in (5.5) above. As in the standard 't Hooft limit, we take $N\to\infty$ while holding fixed the dynamical scale Λ. The new ingredient is that the superpotential couplings are scaled in such a way that the mass of the lightest $\hat{G}$-charged state is held fixed. As always, the mapping to string theory is such that g_s scales like $1/N$. The tension of the confining string is identified with that of the IIB string. We also know that the charged dibaryons and magnetic monopoles of the field theory are identified with D3 branes wrapped on cycles which vanish at the critical point. Thus field theory double-scaling therefore corresponds to a limit in the dual string theory where $g_s\to 0$ with α' fixed, and the moduli of the Calabi-Yau are tuned to their critical values in such a way that the masses of wrapped branes are also held fixed. This is precisely the same type of limit considered by Giveon and Kutasov in the context of the generalised conifold without flux.

We also argued above that only one sector of the field theory, denoted $\mathcal{H}_-$, retains finite interactions in the double-scaling limit, while the remaining states become free. This feature closely resembles the decoupling in the Giveon-Kutasov limit of the interacting LST describing states localised at the singularity from the free theory in the bulk. On the other hand, the double-scaling limit described above keeps Λ fixed, so the size of the non-vanishing cycles in the geometry stays fixed in string units. This means that the dual string background still involves RR field-strengths which remain large in the limit. Despite this we would like to propose that the interacting sector of the four-dimensional field theory is correctly described by the double-scaled Little String Theory of the generalised conifold without flux. For brevity we will refer to the latter theory as DSLST. In the rest of this section, we will explain why such a correspondence should hold.

The first observation is that the massless degrees of freedom in the interacting sector $\mathcal{H}_-$ and their F-term interactions precisely match those of DSLST. In terms of $\mathcal{N}=1$ multiplets, both theories contain $p=(m-1)/2$ U(1) vector multiplets and p neutral massless chiral multiplets. The F-term effective action for these degrees of freedom is given for the field theory in eqs. (5.17), (5.24), (5.25) and for DSLST in eqs. (6.13), (6.14), (6.17). The two sides of the correspondence match exactly if we identify the fields according to,

$$\tilde{w}_{\alpha l} \leftrightarrow \mathcal{W}_{\alpha l} \qquad \tilde{S}_l \leftrightarrow K\varphi_l \tag{6.20}$$

for $l=1,2,\ldots,p$ where $K=2i\Lambda^{\frac{m}{2}}\kappa\sqrt{\alpha'}$ will be identified below. We must also identify the double-scaling parameters on both sides as $\tilde{g}_l \leftrightarrow \tilde{\mu}_l$ for $l=2,3,\ldots,p$. This ensures that the reduced spectral curve Σ_- matches the curve Γ which governs the F-terms of DSLST.

It is straightforward to reinterpret this agreement in terms of the dual string geometry. To do so we rescale the spacetime coordinates in (6.18) as follows,

$$u=\tilde{u}\delta^{\frac{1}{2}}\alpha^{\frac{1}{2}} \qquad v=\tilde{v}\delta^{\frac{1}{2}}\alpha^{\frac{1}{2}} \qquad w=\tilde{w}\delta^{\frac{1}{2}}\alpha^{\frac{1}{2}} \qquad x=\tilde{x}\delta^{\frac{1}{m}} \tag{6.21}$$

where $\delta=g_1-2\Lambda^m$ and $\alpha=-4\varepsilon^2\Lambda^m$. We now take the double-scaling limit (5.5) while keeping the rescaled coordinates fixed. Keeping only the leading terms, (6.18) becomes,

$$\tilde{u}^2+\tilde{v}^2+\tilde{w}^2+\tilde{x}^m-\sum_{l=2}^{p}\tilde{g}_p\tilde{x}^{l-1}-1=0\,. \tag{6.22}$$

The resulting geometry can be thought of as an a_1-fibration of the reduced spectral curve Σ_- and also coincides with the rescaled generalised conifold (6.7) with identification $\tilde{g}_l=\tilde{\mu}_l$.

With the identifications (6.20), we see that the two theories coincide exactly at the level of F-terms. On the other hand, an obvious discrepancy is that the DSLST has $\mathcal{N}=2$ supersymmetry in four dimensions while the field theory apparently has only $\mathcal{N}=1$ SUSY. Note however that the chiral multiplets $\tilde{S}_l$ become exactly massless in the double-scaling limit. Further the superpotential for these degrees of freedom vanishes exactly in this limit. These facts are consistent with the proposal that the supersymmetry of the interacting sector of the field theory is enhanced from $\mathcal{N}=1$ to $\mathcal{N}=2$ in the double-scaling limit. In particular, our large-N counting arguments imply that the glueballs of the interacting sector have couplings of order one to the massive $\hat{G}$-charged states. Normally such interactions would generate non-zero masses for the glueballs unless some symmetry forbids it. In the present case, there are no obvious candidates apart from an enhanced $\mathcal{N}=2$ supersymmetry.

In the dual string theory, the superpotential and the consequent breaking of $\mathcal{N}=2$ SUSY down to $\mathcal{N}=1$ are directly associated with the presence of non-zero RR fluxes. As we have emphasized above, the full string geometry certainly does contain strong RR fluxes. On the other hand, the superpotential for the fields in the interacting sector vanishes in the double-scaling limit suggesting that this sector of the theory is unaffected by the flux. The heuristic picture which reconciles these apparently contradictory features is as follows. The generalised conifold singularity involves the same set of light charged states wrapped on the vanishing three-cycles whether or not there is flux through the remaining non-vanishing cycles. Thus we expect the same infinite throat to form in the region near the singularity in both cases. As discussed in section 5, the degeneration of the spectral curve Σ in the double scaling limit is consistent with two curves Σ_+ and Σ_- joined by a long tube as shown in figure 5. As the dual spacetime geometry is an a_1-fibration of Σ, we are essentially suggesting that the spacetime metric in the directions along Σ is similar to the one implied in figure 5.

Given the presence of a throat region near the origin, the resulting linear growth of the dilaton provides a natural explanation of the breakdown of the $1/N$ expansion in the dual field theory. It also suggests an explanation for the puzzle of the RR flux raised above. Even though RR flux is present in the bulk geometry, as there are no RR charges in the throat, the RR field-strength should decay as we move down the throat. In the limit where the asymptotic string coupling goes to zero, the only states which retain finite interactions are localised infinitely far down the throat. These states should therefore be unaffected by the presence of flux in the bulk. With this interpretation, the rescaled spacetime coordinates defined in (6.21), which are held fixed in the double-scaling limit, correspond to coordinates for the region far down the throat where the effective string coupling remains fixed as $g_s \to 0$. In the next section we will discuss a related model where these points can actually be made quite precise.

To make our proposal more concrete, we should specify the relation between the parameters of the two theories. We begin by considering the $\mathbb{Z}_m$ symmetric theory with $\tilde{g}_l = \tilde{\mu}_l = 0$ The double-scaled Little String Theory of the generalised conifold singularity is characterised by the renormalised string coupling κ and the fundamental string tension $1/2\pi\alpha'$. In field theory, these correspond to the effective three-point glueball coupling and the confining string tension respectively. We will mainly be interested in the case $\kappa \ll 1$, where string perturbation theory is valid. In this case we propose the identifications,

$$\begin{aligned} \kappa = \frac{1}{N_{\text{eff}}} &= \frac{\Lambda^{3-\frac{m}{2}}}{\Delta} F_1(\tilde{\varepsilon}, \Lambda, M_{UV}) \\ \alpha' &= \frac{1}{\Lambda^2} F_2(\tilde{\varepsilon}, \Lambda, M_{UV}) \,, \end{aligned} \tag{6.23}$$

where F_1 and F_2 are two unknown dimensionless functions of the parameters which remain fixed in double scaling limit. These include $\tilde{\varepsilon} = \varepsilon/N$,$\Lambda$, and the UV cut-off M_{UV}. Thus we find the constant K appearing in (6.20) is given as $K = 2i\Lambda^2 F_1 \sqrt{F_2}$. Away from the $\mathbf{Z}_m$ symmetric point the DSLST also has $p-1$ additional parameters $\tilde{\mu}_l$ associated with the normalisable deformations of the geometry. As above, comparison of F-term observables implies that these are identified with the corresponding field theory parameters as $\tilde{g}_l = \tilde{\mu}_l$ for $l = 2, 3, \ldots, p$.

7. The β-deformed theory and its AdS dual

As emphasized in the introduction, partially confining phases of the type studied above are generic to $\mathcal{N} = 1$ supersymmetric theories with adjoint matter. In this section we will study a very special model in this class: the β-deformation of $\mathcal{N} = 4$ SUSY Yang-Mills. The matter content of this model is an $\mathcal{N} = 1$ vector multiplet W_α together with three adjoint chiral superfields, Φ_1, Φ_2 and Φ_3 with superpotential,

$$W = \mathrm{Tr}_N \left[e^{+i\beta/2} \Phi_1 \Phi_2 \Phi_3 - e^{-i\beta/2} \Phi_1 \Phi_3 \Phi_2 \right]. \tag{7.1}$$

We will consider the case of gauge group $G = \mathrm{SU}(N)$. This theory is distinguished because it corresponds to an exactly marginal deformation of $\mathcal{N} = 4$ SUSY Yang-Mills [51] and inherits some of special properties of the latter theory. In particular as the deformation is marginal the resulting theory shares the exact conformal invariance of $\mathcal{N} = 4$ SUSY Yang-Mills. It also inherits an exact electric-magnetic duality related to the S-duality of the $\mathcal{N} = 4$ theory [52]. Finally, proximity to $\mathcal{N} = 4$ SUSY Yang-Mills means that the deformed theory can easily be studied in the context of the AdS/CFT correspondence using IIB supergravity.

Despite these special features we will see that the β-deformed theory has very similar behaviour to the model (2.1). In particular it has a partially confining branch which exhibits many of the features we discussed above, including a critical point where charged states become massless. The key point is that, for this particular model, the physics of the critical point can be studied quantitatively using the AdS/CFT correspondence. This section is mainly a review of the results of [26, 27], interpreted in the context of our more general study.

The model with superpotential (7.1) corresponds to a two complex parameter family of conformal field theories labelled by the complexified gauge coupling $\tau = 4\pi i/g^2 + \theta/2\pi$ and the deformation parameter β. The $\mathcal{N} = 4$ theory with gauge group $\mathrm{SU}(N)$ is obtained by setting $\beta = 0$. Turning on the deformation breaks the $\mathcal{N} = 4$ superconformal invariance down to an $\mathcal{N} = 1$ sub-algebra. At linear order, the β-deformation corresponds to adding a certain chiral primary operator to the $\mathcal{N} = 4$ Lagrangian. Via the AdS/CFT correspondence, the deforming operator is mapped to a dual supergravity field in AdS_5. The field in question is a particular mode of the complexified three-form field-strength, $G^{(3)} = F^{(3)}_{RR} + \tau H^{(3)}_{NS}$, of IIB supergravity compactified on S^5. Starting from the string dual of the $\mathcal{N} = 4$ theory on $AdS_5 \times S^5$, we can introduce the β-deformation by turning on an appropriate source for $G^{(3)} \sim \beta$ on the boundary of AdS_5. As the deformed theory is conformally invariant, the resulting string background has the form $AdS_5 \times \tilde{S}_5$, where $\tilde{S}_5$ is a deformation of the five-sphere. The corresponding geometry was constructed to second order in a perturbation series in β in [17]. More recently an exact supergravity solution, valid for all values of β, has been obtained by Lunin and Maldacena [18].

A surprising aspect of the β-deformed theory is that it has a much richer vacuum structure than that of the $\mathcal{N} = 4$ theory. The latter theory has a conformal invariant phase and also a Coulomb branch where the gauge symmetry is spontaneously broken to its Cartan subalgebra. For generic values of β, the deformed theory also has a conformal point and a Coulomb branch. However, for special values of β, it also has branches of vacua in various Higgs, confining and oblique confining phases. These branches were studied in detail in [26 − 28] by solving the corresponding Dijkgraaf-Vafa matrix model in the planar limit. In the following we will focus on a particular branch of vacua which exhibits partial confinement of the type discussed in the previous section. In particular we will review several results from [27] and refer the reader to this paper for further details.

We will consider the β-deformed theory with gauge group $G = \mathrm{SU}(N)$ where $N = mn$ for some integers m and n and, for simplicity, we set the vacuum angle to zero so that $\tau = 4\pi i/g^2$. If we choose the special value $\beta = 8\pi^2 i/g^2 n$, the theory has a branch of vacua with low-energy gauge group $\hat{G} = \mathrm{U}(1)^{m-1}$. Along this branch, the lowest components of the three chiral superfields Φ_i acquire non-zero vacuum expectation values (VEVs). A gauge invariant description of these VEVs is,

$$\begin{aligned} \frac{1}{N}\left\langle \mathrm{Tr}_N \Phi_i^k \right\rangle &= \beta_i^{(l)} \qquad \text{for } k = ln \\ &= 0 \qquad\quad \text{otherwise} \end{aligned}$$

The resulting branch is parametrised by $3m$ complex moduli $\beta_i^{(l)}$, with $l = 1, 2, \ldots, m$ and $i = 1, 2, 3$. For generic points on this branch the low-energy gauge group is $\hat{G} = \mathrm{U}(1)^{m-1}$ and the massless spectrum includes $m - 1$ abelian vector multiplets and $3m$ $\hat{G}$-neutral chiral multiplets corresponding to fluctuation of the moduli $\beta_i^{(l)}$.

The branch described above exhibits partial confinement of the original gauge group, $G = \mathrm{SU}(N)$, down to the low-energy gauge group $\hat{G} = \mathrm{U}(1)^{m-1}$. More precisely, we will see that the confinement index, t, introduced above takes the value n. This result can be established in several different ways. As in [26], we may use the exact S-duality of the theory to map the confining branch at $\beta = 8\pi^2 i/g^2 n$ to a Higgs branch which occurs for $\beta = 2\pi/n$. The dual phenomenon, in which magnetic charges are partially confined by the formation of flux tubes can then be studied using semiclassical methods. An alternative approach, developed in [28], is to find the root of the confining branch using an exact description of the Coulomb branch as the moduli space of a certain complex curve (see also [53]). The root of the new branch occurs at a special point on the Coulomb branch where additional magnetic degrees of freedom become massless. These degrees of freedom condense on the new branch leading to partial confinement of electric charges. As usual, confinement can be demonstrated explicitly using an abelian effective Lagrangian which is valid near the root. Finally, as we now review, the physics of the confining branch can also be analysed using the AdS/CFT correspondence [27].

As mentioned above the AdS-dual of the β-deformed theory in its conformal phase is a spacetime of the form $AdS_5 \times \tilde{S}^5$ with a source for $G^{(3)} \sim \beta$ on the boundary of S^5. Here we have the usual identifications $g^2 = 4\pi g_s \sim 1/N$ and $\lambda = g^2 N = \mathcal{R}^4/\alpha'^2$ where $\mathcal{R}$ is the radius of the geometry. The supergravity description is valid provided $\lambda = g^2 N \gg 1$. Choosing the slightly stronger condition $g^2 n \gg 1$, we find that the special value $\beta = 8\pi^2 i/g^2 n$ can be made small and the dual background can be thought of as a small perturbation of $AdS_5 \times S^5$. Under these conditions, a string dual for the new branch was identified in [27]. Specifically a generic vacuum on the partially confining branch is dual to a configuration of m NS fivebranes in[17] $AdS_5 \times S^5$ with the following properties:

1. Each fivebrane is located at a definite radial position $r = \bar{r}_l$ in AdS_5, for $l = 1, 2, \ldots, m$, and fills the four dimensions parallel to the boundary. The radii $\bar{r}_l$ are adjustable moduli of the configuration determined by the field theory VEVs.[18] Without loss of generality we can choose $\bar{r}_1 \leq \bar{r}_2 \leq \ldots \leq \bar{r}_l$.

2. Each fivebrane is wrapped on a toroidal subspace of S^5, $T^2_{(l)} \subset S^5$. The remaining moduli of the configuration correspond to the shape of each torus and its orientation on S^5.

3. Each fivebrane carries exactly n units of D3 brane charge, which can be realised as n units of magnetic flux of the six-dimensional world-volume gauge field through the torus $T^2_{(l)}$.

The NS fivebrane configuration is stabilised by the presence of a non-zero field strength for $H^{(3)}_{NS} \sim 1/n$ corresponding to $\beta = 8\pi^2 i/g^2 n$. The Neveu-Schwarz threeform couples magnetically to the NS5 branes and renders the toroidally wrapped branes marginally stable. This is a version of Myers dielectric effect [54] which arises in the analysis of $\mathcal{N} = 1^*$ SUSY Yang-Mills [29].

Several features of the partially confining branch can be understood in terms of the brane configuration described above. First, the low energy gauge group of the dual field theory is visible in the worldvolume theory on the NS5-branes. In particular, each fivebrane has an abelian gauge field living on its six-dimensional worldvolume. After dimensional reduction on the torus $T^2_{(l)}$, we obtain a U(1) gauge field in the four non-compact dimensions on each brane. As in [29], we expect that the central U(1) subgroup of $\mathrm{U}(1)^m$ is lifted by its coupling to the bulk fields, leaving the low-energy gauge group $\hat{G} = \mathrm{U}(1)^{m-1}$.

To analyse the confinement of electric charges, we introduce external charges in the $\mathbf{N}$ and $\bar{\mathbf{N}}$ representations of SU(N), separated by a distance R. By standard arguments, these charges correspond to the endpoints of a fundamental string, separated by the same distance R, on the boundary of AdS_5. In a conformally-invariant vacuum, the energetically preferred configuration is one where the string droops into the interior of AdS_5, corresponding to the spreading out of electric flux lines in the dual gauge theory. The result is a Coulomb potential between the charges, $V(R) \sim 1/R$, as dictated by conformal invariance [55].

[17] At lowest order in β we can neglect the deformation of S^5

[18] The precise relation between the radii $\bar{r}_l$ and the moduli $\beta_i^{(l)}$ can be deduced from the results of [27] but will not be needed here.

As explained in [29], the situation is modified by the presence of NS5 branes in the bulk. A key fact about Neveu-Schwarz fivebranes is that fundamental strings cannot end on them. Instead the fundamental string can form a boundstate with the NS5, corresponding to a solitonic string of finite tension σ in its worldvolume gauge theory. The tension of these objects is set by the scale of the VEVs and obeys the standard 't Hooft scaling, $\sigma \sim N^0$. In the present case the string can droop into the bulk geometry as far as one of the NS fivebranes and smoothly join onto a segment of solitonic strings which extends a distance R parallel to the boundary. The result is the onset of a confining linear potential $V(R) \sim \sigma R$ between the external charges in the gauge theory at large distances. However, this is not quite the whole story: the theory also contains m distinct baryon vertices, labelled by an integer $l = 1, 2, \ldots, m$, each corresponding to a D3 brane whose spatial worldvolume fills a three-manifold $M_3^{(l)}$ whose boundary is the torus $T^2_{(l)}$ on which the l'th NS5 is wrapped. As each NS5 brane carries n units of worldvolume magnetic through the torus there there is a non-trivial selection rule [29] which demands that n fundamental strings must end on the D3 brane. The resulting configuration therefore corresponds to a baryon vertex which can screen external charges in the n-fold product of the fundamental representation of $\mathrm{SU}(N)$. Thus we conclude that the confinement index t takes the value $t = n$.

The AdS analysis described above corresponds to a 't Hooft limit of the β-deformed theory where $N = mn \to \infty$ with m, $\lambda = g^2 N$ and the moduli $\beta_i^{(l)}$ held fixed. This is the same type of large-N limit considered in section 3 above, and it is interesting to compare the spectrum of the β-deformed theory with the general discussion given there. From the dual AdS description, we find the following states,

1. The massless spectrum includes $m-1$ photons and their $\mathcal{N}=1$ superpartners as well as 3m additional neutral chiral multiplets.

2. As we have gauge theory flux tubes of tension $\sigma \sim N^0$, we expect to find a spectrum of glueballs with masses of order N^0 corresponding to the excitation spectrum of the corresponding closed strings. These states are neutral under the low-energy gauge group.

3. It is straightforward to identify dibaryonic states carrying electric charges $(+n, -n)$ with respect to $\mathrm{U}(1)_r \times \mathrm{U}(1)_s \subset \hat{G}$. These states correspond to D3 branes wrapped on 3-chains C_{rs} of topology $T^2 \times I$ where I is a closed interval. The two boundaries of the chain are the torii $T^2_{(r)}$ and $T^2_{(s)}$ on which the r'th and s'th NS fivebranes are wrapped. The masses of these states go like $\mathrm{Vol}(B_{rs})/g_s\alpha'^2$. The inverse power of g_s implies that the masses of these states scale like N as expected for baryons.

4. States carrying magnetic charges $(+1, -1)$ under $\mathrm{U}(1)_r \times \mathrm{U}(1)_s \subset \hat{G}$ correspond to D-strings stretched between the r'th and s'th NS fivebranes with masses of order $D_{rs}/g_s\alpha'$ where D_{rs} is the proper distance between the branes. As before, the inverse power of g_s implies that the masses of these states scale like N. These states correspond to unconfined magnetic monopoles.

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Thus, the large-N spectrum is in qualitative agreement with the field theory analysis of section 2. The only significant difference is the presence of massless chiral multiplets in generic vacua. This reflect the conformal invariance of the theory. Indeed conformal invariance and other global symmetries are spontaneously broken on the partially confining branch and some of the massless scalars living in the chiral multiplets can be interpreted as Goldstone bosons.

7.1 The critical point

So far we have discussed generic points on the partially confining branch of the β-deformed theory. The string dual to these vacua is a configuration of m NS fivebranes at distinct radial positions $r = \bar{r}_l$ in AdS_5, for $l = 1, 2, \ldots, m$ wrapped on distinct tori $T^2_{(l)}$. In this case, the spectrum of the theory includes massive states corresponding to D-strings and toroidally wrapped D3 branes stretched between the NS five-branes. In the previous section, we identified these states as magnetic and electric dibaryons respectively. In generic vacua, the masses of these states scale like $g_s^{-1} \sim N$. However, there are also special points in the moduli space where two or more NS5 branes coincide and some of these states become massless. In this subsection we will discuss interpretation of these special points in, both in field theory and in the string dual.

We will focus on the maximal critical point where all m NS-fivebranes become coincident. The new massless states include $m(m-1)$ stretched D-strings and, as for NS fivebranes in flat space, we expect the low-energy gauge group to be enhanced from $\hat{G} = \mathrm{U}(1)^{m-1}$ to $\hat{H} = \mathrm{SU}(m)$. We also find $m(m-1)$ new massless states corresponding to stretched D3 branes which are magnetically charged under $\hat{G}$. As both electric and magnetic charges are becoming light simultaneously, we expect the theory in these special vacua to flow to an interacting CFT in the IR. For m coincident toroidally-wrapped NS5 branes in flat space, the resulting CFT would be $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\hat{G} = \mathrm{SU}(m)$. As shown in [27], the $\mathcal{N} = 4$ theory also arises in the present case where the fivebranes live in an asymptotically-AdS space.

In the dual field theory, the corresponding submanifold of the partially confining branch is parametrised by three complex numbers α_1, α_2 and α_3, defined by

$$\begin{aligned} \frac{1}{N}\left\langle \mathrm{Tr}_N \Phi_i^k \right\rangle &= \alpha_i^k \qquad \text{for } k = ln \\ &= 0 \qquad\ \ \text{otherwise} \end{aligned}$$

Equivalently, in terms of the moduli $\beta_i^{(l)}$ introduced above we have $\beta_i^{(l)} = \alpha_i^{ln}$. As before, these vacua only occur for the special value $\beta = 8\pi^2 i/g^2 n$. The low energy effective theory in these vacua is $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\hat{G} = \mathrm{SU}(m)$ and gauge coupling $\hat{g}^2 = g^2 n$. These conclusions can be checked by using the exact description of the vacuum structure of the theory presented in [28], to verify the low energy gauge group, gauge coupling and moduli space near the corresponding root.

A notable feature of the above results is that the rank m of the IR gauge group and the corresponding gauge coupling $\hat{g}^2 = 16\pi^2/g^2 n$ remain fixed when we take the 't Hooft limit of the $\mathrm{U}(N)$ theory. To understand the implications of this, it is useful to consider the correlation functions of single-trace operators. We will focus on L-point correlation function of the operator $\hat{O} = \mathrm{Tr}_N[F_{\mu\nu}F^{\mu\nu}]$ appearing in (3.1). The large-distance behaviour of this correlation function will be dominated by the fluctuations of the unconfined components of the gauge field. In this case the non-abelian field strength $F_{\mu\nu} \in \mathrm{Lie}(G)$ can be replaced by its unconfined component $f_{\mu\nu} \in \mathrm{Lie}(\hat{G})$ inside the trace and the resulting correlations calculated in the low-energy effective theory.

For generic vacua on the partially confining branch we have $\hat{G} = \mathrm{U}(1)^m$. The low-energy theory is abelian and hence free, implying that all connected correlation functions apart from the two-point function vanish. The normalization of the two-point function is determined by the low-energy gauge coupling $\hat{g}^2 = 16\pi^2/g^2 n$. The resulting correlations functions trivially obey the bound,

$$\langle \hat{O}(x_1)\hat{O}(x_2)\ldots\ldots\hat{O}(x_L)\rangle \leq N^{2-L} \tag{7.2}$$

implied by standard large-N scaling. Now consider the theory at the critical point, which has SU(m) gauge interactions with a coupling which remains fixed in the large-N. Perturbation theory in $\hat{g}^2$ yields contributions limit to L point functions of order N^0 for any L. Thus the bound (7.2) is badly violated at the critical point. This indicates that the 't Hooft large-N expansion considered above breaks down at the critical point.

7.2 The gravity dual

The string dual of the vacuum (7.2) was studied in detail in [27]. The relations between string and gauge theory parameters are simplest when one of the three moduli (say α_3) is set to zero. In this case the dual configuration consists of m-coincident NS fivebranes located at fixed radial distance $r = \bar{r} = \sqrt{r_1^2 + r_2^2}$ with $r_i = |\alpha_i|(2\pi\alpha')$ for $i = 1, 2$. The NS5 branes are wrapped on a rectangular torus $T^2 \subset \tilde{S}^5$ and carry a total of $N = mn$ units of D3 brane charge.

Following Polchinski and Strassler [29], one can find a smooth supergravity solution corresponding to the brane configuration described above which is valid provided $g^2 n \gg m$. The resulting geometry interpolates between two regions:

Region I: sufficiently far away from the branes, the effect of N units of D3 branes charge dominates over that of the NS5 branes. Specifically, for $v = |r - \bar{r}| \gg v_{cr}$ with,

$$v_{cr} = \sqrt{\frac{m r_1 r_2}{g_s n}} \tag{7.3}$$

the dual geometry can be approximated by the one produced by the toroidal distribution of N D3 branes only. This leads to a warped AdS_5 metric corresponding to a point on the Coulomb branch of $\mathcal{N} = 4$ SUSY Yang-Mills.

Region II: very close to the toroidally wrapped branes, the gravitational effect of the NS5 branes dominates over that of the D3 branes. The dual geometry can be approximated by the the near-horizon geometry of m NS fivebranes compactified to four dimensions on a torus. This is valid for $v \ll v_{cr}$.

The full supergravity solution interpolating between these two regions was given in [27]. For the present purposes we will only need the metric in region II, which can be written in the form,

$$ds^2 = \eta_{AB} dY^A dY^B + d\sigma^2 + \alpha' m d\Omega_3^2 \,. \tag{7.4}$$

This is the standard infinite throat geometry of the NS fivebrane. The Y_A, with $A = 0, 1, 2, \ldots 5$ are coordinate on the brane world-volume and η_{AB} is the flat metric on six dimensional Minkowski space. Compactification to four dimensions on T^2 is implimented by the identifications,

$$Y_4 \sim Y_4 + 2\pi L_1 \qquad Y_5 \sim Y_5 + 2\pi L_2$$

with

$$L_1 = \left(\frac{g_s n \alpha'}{r_1 r_2}\right)^{\frac{1}{2}} r_1 \qquad L_2 = \left(\frac{g_s n \alpha'}{r_1 r_2}\right)^{\frac{1}{2}} r_2 \,.$$

It is important to note that that energies of string states in the geometry (7.4) are measured relative to the time coordinate Y_0 on the fivebrane worldvolume. On the other hand energies in gauge theory are naturally measured relative to the time coordinate x_0 on the boundary of AdS_5. The relation between Y_0 and x_0 is fixed uniquely by matching the solutions in region I and region II [27]. The result is, $Y_0 = (T_{\rm IIB}/T_{\rm QFT})^{-\frac{1}{2}} x_0$ where,

$$T_{\rm IIB} = \frac{1}{2\pi\alpha'} \qquad T_{\rm QFT} = \frac{8\pi^2 |\alpha_1||\alpha_2|}{g^2 n} \,.$$

The upshot of this rescaling is that the fundamental IIB string of tension $T_{\rm IIB}$, propagating in the throat geometry (7.4) is identified with a gauge theory flux tube of tension $T_{\rm QFT}$. The above formula for the gauge theory flux tube tension can also be checked explicitly via S-duality to the Higgs branch where the corresponding object emerges as a classical soliton.

The four directions transverse to the brane are labelled by a radial coordinate $\sigma = -\sqrt{m\alpha'} \log(g_s v_{cr}/v)$ and three polar angles on the 3-sphere. The IIB dilaton ϕ grows linearly along the radial direction,

$$\phi = -\frac{Q}{2}\sigma \qquad Q = \frac{2}{\sqrt{m\alpha'}} \,.$$

Hence the effective string coupling $g_s^{\rm eff} = \exp(\phi)$ grows in the direction of increasing σ, becoming equal to unity at $\sigma = 0$. Consequently string perturbation theory breaks down for states propagating in the region $\sigma > 0$. This corresponds to the breakdown of the $1/N$ expansion in the dual field theory discussed in the previous section.

As we have m NS5 branes the solution also includes m units of Neveu-Schwarz three-form flux through the transverse S^3. However, note that the region II solution, valid for $v \ll v_{cr}$, contains no background Ramond-Ramond fields. The full geometry contains both three-form and five-form RR field-strengths, but these die off rapidly as we enter the five-brane throat [27]. The throat geometry (7.4) corresponds to an exact solution of IIB string theory and has a worldsheet description [56] where the radial direction is realised as a free scalar with an appropriate background charge and the S^3 factor becomes an SU(2) WZW model of level m. This description is useful for analysing states propagating in the region where the effective string coupling is small.

7.3 The double-scaling limit

The near-horizon geometry of parallel NS fivebranes is the throat geometry described in the previous section. String theory in this background is hard to analyse because of the growth of the dilaton. As discussed in the introduction above, we can cure this problem by defining an appropriate double-scaling limit.

Following Giveon and Kutasov [8], we consider a configuration where the NS5 branes are no longer coincident. Seperating the NS5 branes corresponds to moving away from the critical point on the partially confining branch. In particular, we can produce a circular distribution of m NS fivebranes by introducing a VEV for the chiral superfield Φ_3,

$$\frac{1}{N}\langle \mathrm{Tr}_N \Phi_3^N \rangle = \bar{\mu}^N \, . \tag{7.5}$$

The VEVs of Φ_1 and Φ_2 are given in terms of moduli α_1 and α_2 as in (7.2). As we will eventually take a limit where $\bar{\mu} \to 0$, we will choose $|\bar{\mu}| \ll |\alpha_1|$, $|\alpha_2|$.

As we move away from the critical point we find a spectrum of massive states corresponding to stretched D-strings. As the NS5 branes are wrapped on tori, we also find states corresponding to stretched wrapped D3 branes. As explained above, these states are identified with magnetic and electric dibaryons respectively. The deformation can also be understood as moving onto the Coulomb branch of the low-energy theory which is $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group SU(m) and gauge coupling $\hat{g}^2 = g^2 n$. This low energy theory has an S-dual formulation with coupling $16\pi^2/\hat{g}^2 = 16\pi^2/g^2 n$. In the latter formulation, the electric dibaryons are realised as BPS saturated W-bosons while the magnetic dibaryons are BPS monopoles. There will also be dyonic dibaryons carrying both electric and magnetic charges. The S-duality of the low-energy theory permutes these states.

The double scaling limit involves taking g_s to zero while holding the masses of stretched D-strings and the fundamental string tension fixed. In field theory this means taking $N \to \infty$ and $\bar{\mu} \to 0$ with $\lambda = g^2 N$ and $m = N/n$ held fixed. In addition we also hold the moduli α_1, α_2 and $\rho = \bar{\mu} N$ fixed. In this limit the Region II solution becomes the cigar geometry of Giveon and Kutasov, with two directions compactified on a torus. This corresponds to IIB string theory on,

$$R^{3,1} \times T^2 \times \left(\frac{\mathrm{SL}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \right) / \mathbb{Z}_m \, , \tag{7.6}$$

where both cosets are at level m as above. The effective string tension T_{QFT} is the tension of the gauge theory flux tubes studied in [26, 27]. The maximum value, g_{cigar} of the effective string coupling $g_s^{\mathrm{eff}} = \exp(\phi)$ is attained at the tip of the cigar. In terms of gauge theory parameters we have,

$$T_{\mathrm{QFT}} = \frac{8\pi^2 m |\alpha_1||\alpha_2|}{\lambda} \qquad g_{\mathrm{cigar}} \sim \frac{2\pi}{\rho} \sqrt{\lambda m |\alpha_1||\alpha_2|} \, . \tag{7.7}$$

The radii of the torus are,

$$R_1 = \frac{\lambda}{8\pi^2 m |\alpha_1|} \qquad R_2 = \frac{\lambda}{8\pi^2 m |\alpha_2|} \, . \tag{7.8}$$

For large but finite N, the dual string background consists of the asymptotically-AdS Region I and the cigar-like Region II described above. The two regions are joined by a tube or throat of proper length $L \sim \log N$. The Hilbert space of the dual theory has two sectors, one corresponding to string states localised in Region I which have interactions controlled by $1/N \sim g_s$ and the other to states localised near the tip of the cigar which have interactions controlled by $1/N_{\rm eff} \sim g_{\rm cigar}$. In the large-$N$ limit these two sectors decouple as discussed in the Introduction.

7.4 Summary

To conclude this section we will sumarise the common features of the two models studied in detail in this paper. For brevity we will refer to the model with superpotential (2.1) as Model A and the β-deformed theory as Model B.

1. Model A (B) has a phase where the gauge group $G = \mathrm{U}(N)$ ($\mathrm{SU}(N)$) is partially confined down to an abelian subgroup $\hat{G} = \mathrm{U}(1)^m$ ($\mathrm{U}(1)^{m-1}$) with $N = mn$. The screening of external charges is characterised by the existence of m $(m-1)$ different baryon vertices and the confinement index of [25] takes the value $t = n$ in the relevant vacua of both models.

2. In generic vacua both models have a standard 't Hooft limit where $N = mn \to \infty$ with m fixed leading to a weakly coupled closed string dual. The large-N spectrum includes states carrying electric and magnetic charges under $\hat{G}$. The masses of these states scale linearly with N. In the dual string theory, these states are realised as wrapped (stretched) D-branes for Model A (B).

3. Both models have critical points where electric and magnetic charges simultaneously become massless leading to an interacting conformal field theory. Model B at its critical point flows to $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\mathrm{SU}(m)$ in the IR. It has been proposed in [43] that, for the critical values of the parameters, Model A flows to the Argyres-Douglas $\mathcal{N} = 2$superconformal fixed point of type a_{m-1} in the IR.

4. In both models, the critical point corresponds to the appearance of a singularity in the dual string background. For Model A, the vanishing of certain intersecting three-cycles in a Calabi-Yau three-fold leading to the generalised conifold singularity. For Model B the singularity involved coincident NS5 branes. In both cases we argued that standard $1/N$ expansion breaks down at the critical point.

We also found some differences between the two theories. In particular, Model B has some additional special features due to its close relation to $\mathcal{N} = 4$ SUSY Yang-Mills. The partially confining phase of Model B is realised on a branch with moduli corresponding to the expectation values of massless scalar fields. In contrast for Model A we found isolated vacua. This distinction is related to the underlying superconformal invariance of Model B which could be removed by adding further terms to the superpotential.

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In our study of Model A, the critical point lead to singular behaviour in the low-energy effective action corresponding to the divergence of the sum of planar diagrams. For Model B, the corresponding terms in the F-term effective action actually vanish identically because of enehanced $\mathcal{N} = 4$ SUSY in the IR (which is already present at generic points on the partially confining branch). For example, the exact low-energy gauge coupling is τ/n at all points on the partially confining branch and receives no corrections from planar diagrams. On the other hand, in one corner of the Model B parameter space, we can study the critical point directly using Type IIB supergravity.

Although we studied the two models using very different methods, we were lead to a very similar picture of the critical point in both cases. The main points are,

1. In both models we propose that the region near the singularity in the spacetime of the dual string theory is replaced by an infinite throat with a linear dilaton. The growth of the dilaton corresponds directly to the breakdown of the $1/N$ expansion in the dual field theory.

2. In both models we defined a double-scaling limit where $N \to \infty$ and the parameters/moduli of the theory are tuned to their critical values in such a way that the masses M_B of the $\hat{G}$-charged states are held fixed. As in the conventional 't Hooft limit, the tension, T, of the confining string is also held fixed. The characteristic feature of the double-scaled theory in both cases was the emergence of a decoupled sector with interactions controlled by a parameter $1/N_{\mathrm{eff}} \sim \sqrt{T}/M_B$ which is held fixed in the limit.

3. In both cases we have proposed that the decoupled sector is described by a double-scaled Little String Theory (DSLST). For Model A the relevant DSLST lives in four-dimensions and has $\mathcal{N} = 2$ supersymmetry. For Model B we obtained a description in terms of a six-dimensional DSLST with sixteen supercharges compactified to four-dimensions on a torus.

4. In both cases there is an equivalent description of the DSLST in terms of IIB string theory on a particular linear dilaton background including a semi-infinite cigar. The background is pure NS and the effective string coupling is controlled by the field theory parameter $1/N_{\mathrm{eff}}$

8. Discussion

In this paper we have presented two examples of a new type of duality between double-scaled confining field theories and non-critical superstrings. The main result is a limit in which the spectrum and S-matrix of the interacting sector of the field theory can be calculated using standard worldsheet methods. In particular, the glueball spectrum of the field theory is identified with the discrete spectrum of string states localised near the tip of the cigar in the dual string background. As discussed in [27], the pre-exisiting results [8, 30] on the string side indicate some of the expected features of large-N gauge theory such as a Hagedorn density of states and an S-matrix with linear Regge trajectories. Other features such as the continuous spectrum of plane-wave normalisable states [8] are unexpected[19] but not contradictory: the continuum should be replaced by a discrete spectrum at finite N, with mass splittings which go to zero as $N \to \infty$ [27].

One of the most interesting aspects of the results described above is the close relation between the four-dimensional double-scaling limit and the related phenomena in zero-dimensional matrix models. A key feature of the zero-dimensional case is the emergence of universality classes associated with particular critical points. In that context, the Feynman diagrams of the matrix model correspond to a triangulation of the string world-sheet and the universal behaviour is obtained in the continuum limit of this discrete world-sheet theory. In four-dimensional theories considered here a similar type of universality emerges in the double-scaling limit. In particular, the interacting double-scaled theory only depends on the behaviour near the singularity of the dual string background. Many details of the microscopic theory reside in the bulk region which decouples from the theory in the throat in the double-scaling limit. In this way many different models should yield the same double-scaled theory. The model with superpotential (1.1) provides a striking example of this. This theory is non-renormalisable and our analysis was performed with a fixed UV cut-off M_{UV} in place. Nevertheless, the large-N double-scaling limit yields a continuum theory where all dependence on M_{UV} is absorbed in the definitions (6.23) of the two parameters κ and α'. A decoupled continuum theory with a fixed mass scale, is therefore obtained *without* taking the limit $M_{UV} \to \infty$. A very similar phenomenon was also observed in [27], where the β-deformed model was interpreted as a lattice regularisation of a six-dimensional gauge theory. This type of behaviour is very unexpected from a field theoretic point of view and is probably of more general interest.

The original motivation for this work was to find examples where the 't Hooft large-N limit of a four-dimensional confining field theory is dual to a pure Neveu-Schwarz string background. We were only able to accomplish this at the cost of considering an alternative large-N double-scaling limit. An obvious question is why this was necessary and what is the connection between the double-scaling limit and the absence of RR flux? The close parallels to the double-scaled limit of zero-dimensional matrix models suggest an answer to this question. In this context, the double-scaling limit is a continuum limit on the world-sheet of the string. Before taking this limit we really have a discrete theory where the world-sheet has holes. As discussed in the introduction, this is in line with the emergence of a discrete spin chain description of the $AdS_5 \times S^5$ string [13]. The ideas of [10, 11] also indicate the appearance of holes in the closed string world-sheet due to the presence of background RR flux. In this case, the holes correspond to bubbles of a new phase in a linear σ-model formulation of the string worldsheet theory. Note that, in this formulation, the underlying string σ-model is still formulated on a continuous worldsheet and the holes are described as large fluctuations of the worldsheet fields. There is no contradiction here, as these large fluctuations mean that this formulation inevitably becomes strongly-coupled as soon as a significant number of holes appear. In this case, it is plausible that the appropriate (ie weakly-coupled) effective description is in terms of a discretised world-

[19]See however [58] for earlier discussion of the possibility of a continuous spectrum for large-N QCD.

sheet. If RR fields really lead to a discrete worldsheet theory, then it is natural that the appropriate double-scaling limit, which can be thought of as the corresponding continuum limit, also leads a pure NS background.

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A. F-term couplings in the double-scaling limit

In this appendix, we are going to analyze the double-scaling limit of the couplings $\tilde{V}^{(2)}_{ij}, \tilde{V}^{(3)}_{ijk}$ and the Hessian matrix $\tilde{H}^{(2)}_{ij}$ which were discussed in section 5.

The F-terms of the theory are controlled by the matrix model spectral curve Σ which is given by

$$y^2 = W'(x)^2 - f(x) = \varepsilon^2 \prod_{l=1}^{m} \left(x - x_+^{(l)}\right)\left(x - x_-^{(l)}\right), \tag{A.1}$$

where

$$W'(x) = \varepsilon \left(x^m - \sum_{l=1}^{m} g_l x^{l-1} \right), \qquad f(x) = \sum_{l=1}^{m} \kappa_l x^{l-1}. \tag{A.2}$$

The curve Σ is hyperelliptic, a double cover of the complex plane with $2m$ branch points, $x_\pm^{(l)}, l = 1, \ldots, m$. The Argyres-Douglas singularities correspond to $g_l = 0, \kappa_l = 0, l \geq 2, \kappa_1 = 4\varepsilon^2\Lambda^{2m}, g_1 = \mathcal{U} = \pm 2\Lambda^m$. Let us set

$$x_\pm^{(l)} = e^{\frac{2\pi i l}{m}} \left(\mathcal{U} \pm 2\Lambda^m\right)^{\frac{1}{m}}. \tag{A.3}$$

We are going to focus on the singularity at $\mathcal{U} = 2\Lambda^m$, where the roots $x_-^{(l)}$ collide.

The coupling constants of the low-energy $\mathrm{U}(1)^m$ theory are governed by the spectral curve Σ via the generalized period matrix Π (3.4)(3.5)

$$W^{(2)}_{\mathrm{eff}} = \frac{1}{2} \sum_{k,l=1}^{m} \Pi_{kl}\, \omega_{\alpha k}\, \omega^\alpha_l, \qquad \Pi_{kl} = \frac{\partial S^D_k}{\partial S_l} = \frac{\partial^2 \mathcal{F}}{\partial S_k \partial S_l}, \tag{A.4}$$

$$\int d^2\theta\, W^{(2)}_{\mathrm{eff}} \supset V^{(2)}_{kl} f^k_{\alpha\beta} f^{\alpha\beta\, l}, \qquad V^{(2)}_{kl} = \left\langle \frac{\partial^2 \mathcal{F}}{\partial S_k \partial S_l} \right\rangle. \tag{A.5}$$

In the new basis

$$W^{(2)}_{\mathrm{eff}} = \frac{1}{2} \sum_{k,l=1}^{m} \tilde{\Pi}_{kl}\, \tilde{\omega}_{\alpha k}\, \tilde{\omega}^\alpha_l, \qquad \tilde{\Pi}_{kl} = \frac{\partial \tilde{S}^D_k}{\partial \tilde{S}_l} = \frac{\partial^2 \tilde{\mathcal{F}}}{\partial \tilde{S}_k \partial \tilde{S}_l}. \tag{A.6}$$

The new period matrix $\tilde{\Pi}$ is related to Π as follows. Using

$$\begin{pmatrix} \tilde{B} \\ \tilde{A} \end{pmatrix} = M \cdot \begin{pmatrix} B \\ A \end{pmatrix} = \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix} \qquad M \in \mathrm{Sp}(2m; \mathbb{Z}), \tag{A.7}$$

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we find

$$\tilde{S}_k^D = X_{kl} S_l^D + Y_{kl} S_l \,, \qquad \tilde{S}_k = W_{kl} S_l^D + Z_{kl} S_l \,, \tag{A.8}$$

which in turn implies

$$\tilde{\Pi} = (X\,\Pi + Y)\,(W\Pi + Z)^{-1} \,. \tag{A.9}$$

For instance, the symplectic transformation that connects the basis of cycles shown in figure 3 to the basis shown in figure 4 is given by

$$\begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} . \tag{A.10}$$

However, in order to find the behaviour of $\tilde{\Pi}$ close to the Argyres-Douglas singularity, we will consider

$$M_{ij} = \frac{\partial \tilde{S}_j^D}{\partial \kappa_i} = -\frac{1}{2} \oint_{\tilde{B}_j} \frac{x^{i-1}}{y}\, dx \,, \qquad N_{ij} = \frac{\partial \tilde{S}_j}{\partial \kappa_i} = -\frac{1}{2} \oint_{\tilde{A}_j} \frac{x^{i-1}}{y}\, dx \,, \tag{A.11}$$

which determine the period matrix via

$$\tilde{\Pi}_{ij} = \frac{\partial \tilde{S}_j^D}{\partial \kappa_k} \frac{\partial \kappa_k}{\partial \tilde{S}_i} = \left(N^{-1}\right)_{ik} M_{kj} \,. \tag{A.12}$$

Near the singular point, the curve (A.1) can be approximated by

$$y^2 \approx -2\,\varepsilon^2\, \mathcal{U} \left(x^m - \sum_{l=2}^{m} g_l x^{l-1} - \delta \right) , \qquad \delta = \mathcal{U} - 2\Lambda^m \,. \tag{A.13}$$

Using the above expression, we can easily find the behaviour of M_{ij}, N_{ij} close to the AD point. First of all, let us focus on N_{ij} where $i = \alpha = 1, \ldots, p$, $j = \beta = 1, \ldots, p$. Recalling that $\tilde{A}_\beta = A_-^{(\beta)}$ is a vanishing cycle, we find

$$\begin{aligned} N_{\alpha\beta} &= \frac{\partial S_-^{(\beta)}}{\partial \kappa_\alpha} \sim \int_{x_-^{(l)}}^{x_-^{(l')}} \frac{x^{\alpha-1}}{y}\, dx \sim \frac{1}{\varepsilon\, \mathcal{U}^{1/2}}\, \delta^{\frac{\alpha}{m} - \frac{1}{2}} \int_{\tilde{x}_-^{(l)}}^{\tilde{x}_-^{(l')}} \frac{\tilde{x}^{\alpha-1}}{\tilde{y}}\, d\tilde{x} \\ &\sim \frac{1}{\varepsilon\, \mathcal{U}^{1/2}}\, \delta^{\frac{\alpha}{m} - \frac{1}{2}}\, f_{\alpha\beta}\left(\tilde{g}_l\right) , \end{aligned} \tag{A.14}$$

where we defined

$$x = \delta^{\frac{1}{m}}\, \tilde{x} \,, \qquad \tilde{y}^2 = \tilde{x}^m - \tilde{g}_l \tilde{x}^{l-1} - 1 \,, \qquad \tilde{g}_l = \frac{g_l}{\delta^{\frac{m-l+1}{m}}} \,. \tag{A.15}$$

Similarly

$$M_{\alpha\beta} = \frac{\partial S_{D,-}^{(\beta)}}{\partial \kappa_\alpha} \sim \frac{1}{\varepsilon\, \mathcal{U}^{1/2}}\, \delta^{\frac{\alpha}{m} - \frac{1}{2}}\, f_{\alpha\beta}^D\left(\tilde{g}_l\right) . \tag{A.16}$$

We see that, since $\alpha < m/2$, both $N_{\alpha\beta}$ and $M_{\alpha\beta}$ diverge in the limit $\delta \to 0$. Then, let us consider N_{ij} where $i = A = p+1, \ldots, 2p+1$, $j = \beta = 1, \ldots, p$. In this case

$$N_{A\beta} = \frac{\partial S_-^{(\beta)}}{\partial \kappa_A} \sim \int_{x_-^{(l)}}^{x_-^{(l')}} \frac{x^{A-1}}{y}\, dx \sim \frac{1}{\varepsilon\, \mathcal{U}^{1/2}}\, \delta^{\frac{A}{m}-\frac{1}{2}} \int_{\tilde{x}_-^{(l)}}^{\tilde{x}_-^{(l')}} \frac{\tilde{x}^{A-1}}{\tilde{y}}\, d\tilde{x}$$
$$\sim \frac{1}{\varepsilon\, \mathcal{U}^{1/2}}\, \delta^{\frac{A}{m}-\frac{1}{2}}\, f_{A\beta}\left(\tilde{g}_l\right), \tag{A.17}$$

which vanishes in the limit $\delta \to 0$. The same result holds for $M_{A\beta}$.

The behaviour of the periods along the cycles $\tilde{A}_l, \tilde{B}_l, l > [m/2]$, which correspond to the cycles $A_+, B_+, A_\infty, B_\infty$, is markedly different. Indeed, both $N_{\alpha B}, M_{\alpha B}$ and N_{AB}, M_{AB}, $B = p+1, \ldots, 2p+1$, are finite in the limit $\delta \to 0$ and analytic. One can do a Taylor expansion of the integrand in powers of $\mathcal{U} - 2\Lambda^m, \kappa_l, l \geq 2$ and find that the coefficients are finite. Therefore, the periods along the cycles $\tilde{A}_l, \tilde{B}_l, l > [m/2]$ are manifestly analytic at the Argyres-Douglas point. This is a property of the new basis we will exploit again in the following.

In summary, in the limit $\delta \to 0$, the matrices N and M will have the following block structure

$$N \to \begin{pmatrix} N_{--} & N_{-+}^{(0)} \\ 0 & N_{++}^{(0)} \end{pmatrix} \qquad M \to \begin{pmatrix} M_{--} & M_{-+}^{(0)} \\ 0 & M_{++}^{(0)} \end{pmatrix}, \tag{A.18}$$

where by $-$ or $+$ we denote indices in the ranges $\{1, \ldots, p\}$ and $\{p+1, \ldots, 2p+1\}$ respectively. In order to evaluate the generalized period matrix, we also need the inverse of N, which is given by

$$N^{-1} \to \begin{pmatrix} N_{--}^{-1} & \mathcal{N} \\ 0 & \left(N_{++}^{(0)}\right)^{-1} \end{pmatrix}, \qquad \mathcal{N} = -N_{--}^{-1}\, N_{-+}^{(0)}\, \left(N_{++}^{(0)}\right)^{-1}. \tag{A.19}$$

Finally

$$\tilde{\Pi} = N^{-1}\, M \to \begin{pmatrix} N_{--}^{-1} M_{--} & N_{--}^{-1} M_{-+}^{(0)} + \mathcal{N} M_{++}^{(0)} \\ 0 & \left(N_{++}^{(0)}\right)^{-1} M_{++}^{(0)} \end{pmatrix}. \tag{A.20}$$

Let us look more closely at the structure of each block in the above matrix. First of all, by (A.14)

$$\left(N_{--}\right)_{\alpha\beta}^{-1} \sim \varepsilon\, \mathcal{U}^{1/2}\, \delta^{\frac{1}{2}-\frac{\beta}{m}}\, f_{\alpha\beta}^{-1}\left(\tilde{g}_l\right), \tag{A.21}$$

which, by (A.16), implies

$$\left(N_{--}\right)_{\alpha\beta}^{-1}\, \left(M_{--}\right)_{\beta\gamma} = f_{\alpha\beta}^{-1}\left(\tilde{g}_l\right) f_{\beta\gamma}^{D}\left(\tilde{g}_l\right) = \Pi_{\alpha\gamma}^{-}\left(\tilde{g}_l\right). \tag{A.22}$$

Furthermore, since N_{--}^{-1} vanishes in the limit $\delta \to 0$, we find that

$$N_{--}^{-1} M_{-+}^{(0)} + \mathcal{N} M_{++}^{(0)} = N_{--}^{-1}\, M_{-+}^{(0)} - N_{--}^{-1}\, N_{-+}^{(0)} \left(N_{++}^{(0)}\right)^{-1} M_{++}^{(0)} \to 0. \tag{A.23}$$

Therefore, the period matrix has the following block-diagonal form in the double scaling limit

$$\tilde{\Pi} \to \begin{pmatrix} \Pi^- & 0 \\ 0 & \Pi^+ \end{pmatrix}. \tag{A.24}$$

The upper block Π^- is actually the period matrix of the *reduced spectral curve* Σ_-

$$\tilde{y}^2 = \tilde{x}^m - \sum_2^m \tilde{g}_l \tilde{x}^{l-1} - 1 \,. \tag{A.25}$$

To realize this, note that the cycles $A_-^{(\alpha)}, B_-^{(\alpha)}$ form a standard homology basis for Σ_- and that

$$\frac{\tilde{x}^{\beta-1}}{\tilde{y}} d\tilde{x} \,, \qquad \beta = 1, \dots, p \,, \tag{A.26}$$

span a basis of holomorphic 1-forms on Σ_-. Therefore, by (A.14), (A.16) and (A.22), we can conclude that Π^- is the standard period matrix of Σ_-.

Let us now consider the cubic coupling

$$\begin{aligned} \tilde{V}^{(3)}_{ijk} &= \frac{\partial^3 \tilde{F}}{\partial \tilde{S}_i \partial \tilde{S}_j \partial \tilde{S}_k} \\ &= \frac{\partial \kappa_r}{\partial \tilde{S}_i} \left(\frac{\partial^2 \tilde{S}^D_k}{\partial \kappa_r \partial \kappa_p} - \frac{\partial^2 \tilde{S}_s}{\partial \kappa_r \partial \kappa_p} \frac{\partial \kappa_q}{\partial \tilde{S}_s} \frac{\partial \tilde{S}^D_k}{\partial \kappa_q} \right) \frac{\partial \kappa_p}{\partial \tilde{S}_j} \\ &= N^{-1}_{ir} \left(\frac{\partial^2 \tilde{S}^D_k}{\partial \kappa_r \partial \kappa_p} - \frac{\partial^2 \tilde{S}_s}{\partial \kappa_r \partial \kappa_p} \tilde{\Pi}_{sk} \right) (N^{-1})^T_{pj} \,. \end{aligned} \tag{A.27}$$

Recall that in general $\tilde{V}^{(3)} \sim \varepsilon^{-1}$. Therefore, only terms that are divergent in the limit $\delta \to 0$ can survive in the double scaling limit. On the other hand, such terms can only come from the second derivatives of $\tilde{S}^D_k$ and $\tilde{S}_k$ for $k \le p = [m/2]$, since the period integrals along the cycles $\tilde{A}_k, \tilde{B}_k, k > [m/2]$ are analytic in $\Delta\kappa_1 = \kappa_1 - \kappa_1^{cr}, \kappa_l, l \ge 2$.

Eq. (A.27) can be recast in matrix form

$$\tilde{V}^{(3)}_{ijk} = N^{-1}_{ir} \Delta^k_{rp} \, (N^{-1})^T_{pj} \,. \tag{A.28}$$

In order to estimate the above expression for $k < [m/2]$, we use the fact that

$$\begin{aligned} \Delta^k_{rp} &\sim \oint_{\tilde{A}_k} \frac{x^{r+p-2}}{y^3} \, dx \sim \frac{1}{\varepsilon^3 \, \mathcal{U}^{3/2}} \int_{x_-^{(k)}}^{x_-^{(k')}} \frac{x^{r+p-2}}{\left(\sqrt{x^m - g_l x^{l-1} - \delta} \right)^3} \, dx \\ &\sim \frac{1}{\varepsilon^3 \, \mathcal{U}^{3/2}} \frac{\partial}{\partial \delta} \int_{x_-^{(k)}}^{x_-^{(k')}} \frac{x^{r+p-2}}{\sqrt{x^m - g_l x^{l-1} - \delta}} \, dx \sim \frac{1}{\varepsilon^3 \, \mathcal{U}^{3/2}} \frac{\partial}{\partial \delta} \left(\delta^{\frac{r+p-1}{m} - \frac{1}{2}} F(\tilde{g}_l) \right) , \end{aligned} \tag{A.29}$$

where

$$F(\tilde{g}_l) = \int_{\tilde{x}_-^{(k)}}^{\tilde{x}_-^{(k')}} \frac{\tilde{x}^{r+p-2}}{\sqrt{\tilde{x}^m - \tilde{g}_l \tilde{x}^{l-1} - 1}} \, d\tilde{x} \,, \qquad \tilde{g}_l = \frac{g_l}{\delta^{\frac{m-l+1}{m}}} \,.$$

Then, since $F(\tilde{g}_l)$ is an analytic function of $\tilde{g}_l$ and these parameters are kept finite in the double scaling limit, we find

$$\Delta^k_{rp} \sim \frac{1}{\varepsilon^3 \, \mathcal{U}^{3/2}} \, \delta^{\frac{r+p-1}{m} - \frac{3}{2}} \, G\left(\tilde{g}_l\right) , \tag{A.30}$$

where $G(\tilde{g}_l)$ is analytic as well.

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Then, eq. (A.28) is equivalent to

$$\begin{pmatrix} N_{--}^{-1} & \mathcal{N} \\ 0 & \left(N_{++}^{(0)}\right)^{-1} \end{pmatrix} \begin{pmatrix} \Delta_{--}^{k} & \Delta_{-+}^{k} \\ \Delta_{+-}^{k} & \Delta_{++}^{k} \end{pmatrix} \begin{pmatrix} (N_{--}^{-1})^{T} & 0 \\ \mathcal{N}^{T} & \left(N_{++}^{(0)}\right)^{-T} \end{pmatrix}.$$

A careful inspection shows that the most relevant term is actually

$$N_{--}^{-1}\,\Delta_{--}^{k}\,(N_{--}^{-1})^{T}\,, \tag{A.31}$$

namely, in the double scaling limit

$$N^{-1}\,\Delta^{k}\,(N^{-1})^{T} \sim \begin{pmatrix} N_{--}^{-1}\,\Delta_{--}^{k}\,(N_{--}^{-1})^{T} & 0 \\ 0 & 0 \end{pmatrix}.$$

In fact, using

$$(N_{--})_{\alpha\beta}^{-1} = N_{\alpha\beta}^{-1} \sim \varepsilon\,\mathcal{U}^{1/2}\;\delta^{\frac{1}{2}-\frac{\beta}{m}}$$

we find

$$N_{\alpha\beta}^{-1}\,\Delta_{\beta\gamma}^{k}\,N_{\rho\gamma}^{-1} \sim \frac{1}{\varepsilon\,\mathcal{U}^{1/2}}\,\delta^{\frac{1}{2}-\frac{\beta}{m}}\,\delta^{\frac{\beta+\gamma-1}{m}-\frac{3}{2}}\,\delta^{\frac{1}{2}-\frac{\gamma}{m}} = \frac{1}{\varepsilon\,\mathcal{U}^{1/2}}\,\delta^{-\frac{m+2}{2m}}\,. \tag{A.32}$$

In conclusion, the only surviving terms in the double scaling limit are

$$\tilde{V}_{ijk}^{(3)} \sim \frac{1}{\Lambda^{m/2}\,\Delta} v_{\alpha\beta\gamma}\,(\tilde{g}_l)\,, \tag{A.33}$$

for $i=\alpha=1,\dots,p$, $j=\beta=1,\dots,p$, $k=\gamma=1,\dots,p$.

Finally, let us turn our attention to the Hessian matrix

$$\tilde{H}_{ij}^{(2)} = \frac{\partial^2 W_{\rm eff}^{(0)}}{\partial\tilde{S}_i\partial\tilde{S}_j} \sim N_{ik}^{-1}\left(\frac{\partial^2 W_{\rm eff}^{(0)}}{\partial\kappa_k\partial\kappa_l}\right)(N^{-1})_{lj}^{T} = N_{ik}^{-1}\,\Delta_{kl}^{H}\,(N^{-1})_{lj}^{T}\,. \tag{A.34}$$

The effective superpotential in the new basis reads

$$W_{\rm eff}^{(0)} = \sum_{l=1}^{2p}\left(\tilde{N}_l\frac{\partial\tilde{\mathcal{F}}}{\partial\tilde{S}_l} + 2\pi i\tilde{b}_l\tilde{S}_l\right) + \tilde{N}_{2p+1}\frac{\partial\tilde{\mathcal{F}}}{\partial\tilde{S}_{2p+1}} + 2\pi i(\tilde{\tau}_0+\tilde{b}_{2p+1})\tilde{S}_{2p+1}\,. \tag{A.35}$$

However, since there is no flux through the $A_{-}^{(\alpha)}, B_{-}^{(\alpha)}$ cycles

$$\oint_{A_{-}^{(\alpha)}} T(x)dx = \oint_{B_{-}^{(\alpha)}} T(x)dx = 0\,, \tag{A.36}$$

the expression simplifies to

$$W_{\rm eff}^{(0)} = \sum_{\alpha=1}^{p}\left(\tilde{N}_{\alpha}^{+}\frac{\partial\tilde{\mathcal{F}}}{\partial S_{+}^{(\alpha)}} + 2\pi i\tilde{b}_{\alpha}^{+}\,S_{+}^{(\alpha)}\right) + \tilde{N}_{2p+1}\frac{\partial\tilde{\mathcal{F}}}{\partial\tilde{S}_{2p+1}} + 2\pi i(\tilde{\tau}_0+\tilde{b}_{2p+1})\tilde{S}_{2p+1}\,,$$

which is manifestly analytic at the Argyres-Douglas point. In fact, as we remarked before, $S_{+}^{D,(\alpha)}$, $S_{+}^{(\alpha)}$, $\tilde{S}_{2p+1}^{D}$ and $\tilde{S}_{2p+1}$ are all analytic functions of the moduli κ_l close to the critical point. Therefore, the matrix of second derivatives Δ^{H} will be non-singular

$$\Delta_{kl}^{H} \sim \frac{n}{\varepsilon^{3}\,\mathcal{U}^{3/2}}\,\delta^{0}\,. \tag{A.37}$$

Then, using

$$N^{-1} \to \begin{pmatrix} 0 & 0 \\ 0 & \left(N^{(0)}_{++}\right)^{-1} \end{pmatrix}, \tag{A.38}$$

we find that

$$\tilde{H}^{(2)}_{ij} = \frac{n}{\varepsilon \mathcal{U}^{1/2}} \begin{pmatrix} 0 & 0 \\ 0 & h_{++} \end{pmatrix}. \tag{A.39}$$

In conclusion, the only Hessian matrix elements that survive in the double scaling limit are

$$\tilde{H}^{(2)}_{ij} = \frac{n}{\varepsilon\, \Lambda^{m/2}}\, h^{(2)}_{AB}\,, \tag{A.40}$$

where $i = p + A = p + 1, \ldots, 2p + 1$, $j = p + B = p + 1, \ldots, 2p + 1$.

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